dx in Calculus (1)

Infinitesimal Numbers and Non-Standard Analysis

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Abstract

Back in the 17th century, Newton and Leibniz invented calculus. Leibniz's integration symbol $\int$ and infinitesimal difference dx are still used today, which are defined by "limit" in modern textbooks. Therefore some say that dx is nothing more than a meaningless symbol that is only preserved for the sake of formality. The two sets of beliefs are against each other, while they are both influential in the history of mathematics. In this series, we attempt to explore the soul of calculus and the applications of different viewpoints.

Why calculus exists

Mathematicians have always known that everything in our world isn't as simple and measurable as circles, polygons, and ellipses. With a ruler and sufficient time given, we can be patient with narrowing down the error in measurement. What we just described is called the "method of exhaustion," which was used by Archimedes (Αρχιμήδης ο Συρακούσιος, 3rd century BC), Liu Hui (劉徽, 3rd century), and Seki Takakazu (関 孝和, 1642–1708) to approximate π, and by Babylonians to compute trignometric functions. Ideally, an error is a number that we can always make smaller. Therefore it is infinitely small, or in other words, infinitesimal.

Nothing much could be done until René Descartes (1596–1650) invented analytic geometry in the 17th century. The discussion of infinitesimal numbers was resumed and named as "analysis" or "mathematical analysis." Finally, in the same century, Isaac Newton (O.S. 1642–1726) and Gottfried Leibniz (1646–1716), inspired by mathematicians such as Isaac Barrow (1630–1677), made the calculation doable, and hence they are said to be the "inventor" of calculus.[1]

The infinite and the infinitesimal

Infinitesimal number is as mysterious and attracting as its complement, infinite number. Due to Lazare Carnot (1753–1823), infinitesimal numbers are "not simply any null quantities at all, but rather null quantities assigned by a law of continuity which determines the relationship."

Hence we see two rather philosophical questions to be answered: First, what are these numbers? In other words, what is "infinity," what is "infinitesimal," and what is the difference between infinitesimal and zero? One may be able to imagine that all numbers are represented by points on a line, so that infinity is located at the end of the line, which is out of our sight. However, if infinitesimal numbers are infinitely close to the point representing 0, wouldn't that number simply be 0? Second, how does the arithmetic of these numbers behave? For example, what is infinity minus one, and what is infinity minus infinity? What happens if we sum up infinitely many infinitesimal numbers (i.e., integration)? The theory fails to be persuasive without rigid answer to these questions!

[1] Later, the two accused each other of plagiarism, but that's another story.

Hyperreal numbers

Back when the method of calculus was invented, it was described as a treatment for infinitesimal numbers, and hence has been greatly beyond human's imagination. In the early years of the 19$^{\text{th}}$ century, the concept of "limit" was developed from Newton's theory and proved itself a more structural way to understand calculus. Later in the same century, mathematical logic was developed, infinitesimal number has been forgotten because of its lack of logical support.

In the 1960s, Abraham Robinson (1918–1974) proved that there is an "extension" of real numbers without logical contradiction.[2] The reborn theory of infinitesimal numbers is named "non-standard analysis," for the theory of limit is still, undeniably, the foundation of analysis. In Robinson's work, $\mathbb{R}$ denotes the set of real numbers and $*\mathbb{R}$ denotes the extended set, or the set of "hyperreal numbers."[3] He proved that all statements of first order logic that hold in $\mathbb{R}$ would hold in $*\mathbb{R}$ as well. For example,

- The elementary arithmetical operators can be apply to any two hyperreal numbers: summation, subtraction, multiplication, and division (if the divisor is not 0.)
- For any two different hyperreal numbers, one of them is greater than another. (The law of trichotomy.) Moreover, summation and multiplication with a positive number preserves the order.

Infinite and infinitesimal numbers are legit members in $*\mathbb{R}$. The numbers that are greater than all real numbers are called "positive infinity"; those that are smaller than all real numbers are called "negative infinity"; those that are in between two real numbers are "finite" (including infinitesimal numbers); and those that are smaller than all positive real numbers but also greater than all negative real numbers are called "infinitesimal."

To be precise, there are many different numbers in $*\mathbb{R}$ that are "infinity." Similarly, there are many different numbers that are "infinitesimal." For example, if ω is positive infinite, then some other numbers such as $(\omega - 2)$, $(\omega - 1)$, $(\omega + 1)$, $(\omega + 2)$, 2ω, ... are also positive infinite, while $(1/\omega)$, $(2/\omega)$, ... are positive infinitesimal.

In addition, some intuitive results hold. For example,

- [Positive infinity plus positive infinity] is positive infinity.
- [Finite divided by infinity] is infinitesimal.

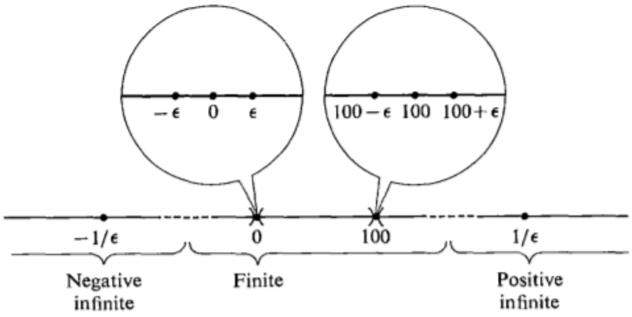
Non-standard analysis

In Newton's and Leibniz's computations, they needed to omit infinitesimal terms at the end. We can explain it with this theorem in $*\mathbb{R}$:

Theorem. *Any finite number x is uniquely represented as a sum of a real number and an infinitesimal number. This real number is denoted as $\operatorname{st}(x)$, read the "standard part" of x.*

[2] With usage of the axiom of choice.

[3] An easier word combination "extended real" is already widely used for the extension with two abstract members $+\infty$ and $-\infty$. There is a total order, but none of the arithmatic operators work well for $\pm\infty$.



The sum of a real number and an infinitesimal number stays close to the real number.
Image source: [5], Section 1.4

With this concept, we define continuous functions, differentiation, and integration as follows.[4]

- If $\mathrm{st}(x) = c$ guarantees $\mathrm{st}(f(x)) = f(c)$, then we say that $f(x)$ is continuous at c.
- Given any infinitesimal number $\mathrm{d}x \neq 0$, the derivative of $f(x)$ at $x = c$ is

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \mathrm{st}\left(\frac{f(c+\mathrm{d}x) - f(c)}{\mathrm{d}x}\right)$$

  (f is differentiable if the result is the same for all choices of such $\mathrm{d}x$.)

- Let ω be an infinite integer[5] and let

$$a = x_0 < \xi_1 < x_1 < \xi_2 < x_2 < \cdots < x_{\omega-1} < \xi_\omega < x_\omega = b$$

  such that $\mathrm{st}(x_n - x_{n-1}) = 0$. Then the integration of $f(x)$ is[6]

$$\int_a^b f(x)\,\mathrm{d}x = \sum_{n=1}^{\omega} f(\xi_n)\,(x_n - x_{n-1})$$

  (All continuous functions are integrable.)

Robinson reinterpreted many existing results in every subjects that uses real numbers.

Comments

Robinson's success undoubtedly represents an interesting viewpoint. Some supporters were even able to simplify the deduction. Of course, the theory attracts criticisms such as usage of the axiom of choice, reliance on a deep understanding of the logic theory, and unconstructability. However, it has never been necessary to beat one aspect by another. Instead, we find and use different aspects to solve problems.

[4] In the first two statements, $f(x)$ is a function defined on an open interval (a, b); in the third statement, $f(x)$ is a function on a closed interval $[a, b]$. $a < c < b$ are three different real numbers.

[5] Extension of "natural numbers." Cf. Section 3.8 in [5].

[6] We extend the "finite summation" function to "infinite summation."

References

[1] Cajori, F.: *A History of Mathematics*. 2nd Ed. Macmillan, New York, 1961.

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[3] Robinson, A.: *Non-Standard Analysis*. North-Holland, Amsterdam, 1966.

[4] Goldblatt, R.: *Lectures on the Hyperreals: An Introduction to Nonstandard Analysis*. Springer-Verlag, New York/Berlin/Heidelberg, 1998.

[5] Keisler, H: *Elementary Calculus: An Infinitesimal Approach*. Made public on http://www.math.wisc.edu/~keisler/calc.html, 2019. Also printed: 3rd Ed. Dover, New York, 2012.

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