Two Simple Proofs for Cramer's Rule

Frank the Giant Bunny

April 9, 2016

Given a non-singular linear system $\boldsymbol{Ax} = \boldsymbol{b}$, *Cramer's rule* states $x_k = \frac{\det \boldsymbol{A}_k}{\det \boldsymbol{A}}$ where $\boldsymbol{A}_k$ is obtained from $\boldsymbol{A}$ by replacing the k^{th} column $\boldsymbol{A}_{*k}$ by $\boldsymbol{b}$; that is,

$$\boldsymbol{A}_k = \left[\boldsymbol{A}_{*1}, \cdots, \boldsymbol{A}_{*k-1}, \boldsymbol{b}, \boldsymbol{A}_{*k+1}, \cdots, \boldsymbol{A}_{*n}\right] = \boldsymbol{A} + (\boldsymbol{b} - \boldsymbol{A}_{*k})\boldsymbol{e}_k^{\intercal} \tag{1}$$

where $\boldsymbol{e}_k$ is the k^{th} unit vector. The proof for Cramer's rule usually begins with writing down the cofactor expansion of $\det \boldsymbol{A}$. This note explains two alternative and simple approaches.

As explained in the page 476 of Meyer's textbook[1], one can exploit the rank-one update form in (1). The *Matrix Determinant Lemma* states that

$$\det(\boldsymbol{A} + \boldsymbol{x}\boldsymbol{y}^{\intercal}) = (1 + \boldsymbol{y}^{\intercal}\boldsymbol{A}^{-1}\boldsymbol{x}) \det \boldsymbol{A}$$

where $\boldsymbol{A}$ is an $n \times n$ non-singular matrix and two vectors $\boldsymbol{x}, \boldsymbol{y}$ are $n \times 1$ column vectors. Then

$$\begin{aligned}
\det \boldsymbol{A}_k &= \det\left(\boldsymbol{A} + (\boldsymbol{b} - \boldsymbol{A}_{*k})\boldsymbol{e}_k^{\intercal}\right) && \text{by definition of } \boldsymbol{A}_k \\
&= \left\{1 + \boldsymbol{e}_k^{\intercal}\boldsymbol{A}^{-1}(\boldsymbol{b} - \boldsymbol{A}_{*k})\right\} \det \boldsymbol{A} && \text{by Matrix Determinant Lemma} \\
&= \left\{1 + \boldsymbol{e}_k^{\intercal}(\boldsymbol{x} - \boldsymbol{e}_k)\right\} \det \boldsymbol{A} && \boldsymbol{Ax} = \boldsymbol{b} \text{ and } \boldsymbol{Ae}_k = \boldsymbol{A}_{*k} \\
&= \left\{1 + (x_k - 1)\right\} \det \boldsymbol{A} && \boldsymbol{e}_k^{\intercal}\boldsymbol{x} = x_k \text{ and } \boldsymbol{e}_k^{\intercal}\boldsymbol{e}_k = 1 \\
&= x_k \det \boldsymbol{A} && \text{by canceling out}
\end{aligned}$$

which completes the proof.

Another simple proof due to Stephen M. Robinson[2] begins by viewing x_k as a determinant

$$x_k = \det \boldsymbol{I}_k = \det\left[\boldsymbol{e}_1 \cdots, \boldsymbol{e}_{k-1}, \boldsymbol{x}, \boldsymbol{e}_{k+1}, \cdots, \boldsymbol{e}_n\right]$$

where $\boldsymbol{I}_k$ is obtained from the identity matrix $\boldsymbol{I}$ by replacing the k^{th} column by $\boldsymbol{x}$. Then $\boldsymbol{AI}_k$ directly yields the matrix $\boldsymbol{A}_k$ in (1) without resort to rank-one update.

$$\begin{aligned}
\boldsymbol{AI}_k &= \boldsymbol{A}\left[\boldsymbol{e}_1 \cdots, \boldsymbol{e}_{k-1}, \boldsymbol{x}, \boldsymbol{e}_{k+1}, \cdots, \boldsymbol{e}_n\right] \\
&= \left[\boldsymbol{Ae}_1 \cdots, \boldsymbol{Ae}_{k-1}, \boldsymbol{Ax}, \boldsymbol{Ae}_{k+1}, \cdots, \boldsymbol{Ae}_n\right] \\
&= \left[\boldsymbol{A}_{*1}, \cdots, \boldsymbol{A}_{*k-1}, \boldsymbol{b}, \boldsymbol{A}_{*k+1}, \cdots, \boldsymbol{A}_{*n}\right] \\
&= \boldsymbol{A}_k
\end{aligned}$$

Then,

$$x_k = \det \boldsymbol{I}_k = \det \boldsymbol{A}^{-1}\boldsymbol{AI}_k = \det \boldsymbol{A}^{-1}\boldsymbol{A}_k = \det \boldsymbol{A}^{-1} \det \boldsymbol{A}_k = \frac{\det \boldsymbol{A}_k}{\det \boldsymbol{A}}$$

which exploits the fact that $\det \boldsymbol{M}^{-1} = 1/\det \boldsymbol{M}$ and $\det \boldsymbol{MN} = \det \boldsymbol{M} \det \boldsymbol{N}$ for two square matrices $\boldsymbol{M}$ and $\boldsymbol{N}$ of the same size.

---

[1] Carl D. Meyer, *Matrix Analysis and Applied Linear Algebra*, SIAM, 2001.

[2] Stephen M. Robinson, "A Short Proof of Cramer's Rule", *Mathematics Magazine*, 43(2), 94–95, 1970.