Solution for Cindy's ODE System

Elizabeth Held

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1 Problem

Solve for $\mathrm{h_a(z_d)}$ and $\mathrm{h_b(z_d)}$ (equations 1 and 2) using the given initial conditions (equations 3 and 4) and continuity conditions (equations 5 and 6).

Specail note: $\mathrm{z_{dr}}$ is $\mathrm{z_d}$' in Cindy's notes.

$$k_z h_a''(z_d) - h_a(z_d)\left(\zeta^2 k_y + p + \omega^2\right) = \sqrt{\frac{2}{\pi}} h_{\mathrm{sD}}\omega \tag{1}$$

$$k_z h_b''(z_d) - h_b(z_d)\left(\zeta^2 k_y + p + \omega^2\right) = \sqrt{\frac{2}{\pi}} h_{\mathrm{sD}}\omega \tag{2}$$

$$k_z h_a'(0) = \gamma h_a(0)(-p) \tag{3}$$

$$h_b'(-1) = 0 \tag{4}$$

$$h_a(z_{\mathrm{dr}}) = h_b(z_{\mathrm{dr}}) \tag{5}$$

$$h_a'(z_{\mathrm{dr}}) - h_b'(z_{\mathrm{dr}}) = \frac{A}{k_z p} \tag{6}$$

2 $h_a(z_d)$ Solution

Equation (7) is $h_a(z_d)$ numerator, equation (8) is $h_a(z_d)$ denominator

$$
\begin{aligned}
&2\sqrt{\frac{2}{\pi}}p\omega h_{\mathrm{sD}}\sqrt{k_z}\left(\gamma p e^{\frac{(2z_d+z_{\mathrm{dr}}+2)\sqrt{\zeta^2k_y+p+\omega^2}}{\sqrt{k_z}}}\right.\\
&+\left(\sqrt{k_z}\sqrt{\zeta^2k_y+p+\omega^2}-\gamma p\right)e^{\frac{(z_d+z_{\mathrm{dr}})\sqrt{\zeta^2k_y+p+\omega^2}}{\sqrt{k_z}}}\\
&-\left(\sqrt{k_z}\sqrt{\zeta^2k_y+p+\omega^2}+\gamma p\right)e^{\frac{(z_d+z_{\mathrm{dr}}+2)\sqrt{\zeta^2k_y+p+\omega^2}}{\sqrt{k_z}}}\\
&\left.+\gamma p e^{\frac{z_{\mathrm{dr}}\sqrt{\zeta^2k_y+p+\omega^2}}{\sqrt{k_z}}}\right)-A\left(e^{\frac{2(z_{\mathrm{dr}}+1)\sqrt{\zeta^2k_y+p+\omega^2}}{\sqrt{k_z}}}\right.\\
&\left.+1\right)\left(\gamma p\sqrt{\zeta^2k_y+p+\omega^2}\left(1-e^{\frac{2z_d\sqrt{\zeta^2k_y+p+\omega^2}}{\sqrt{k_z}}}\right)\right.\\
&\left.+\sqrt{k_z}\left(\zeta^2k_y+p+\omega^2\right)\left(e^{\frac{2z_d\sqrt{\zeta^2k_y+p+\omega^2}}{\sqrt{k_z}}}+1\right)\right)
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
&2p\sqrt{k_z}\sqrt{\zeta^2k_y+p+\omega^2}\left(\gamma p\sqrt{\zeta^2k_y+p+\omega^2}\left(e^{\frac{2\sqrt{\zeta^2k_y+p+\omega^2}}{\sqrt{k_z}}}+1\right)\right.\\
&\left.+\sqrt{k_z}\left(\zeta^2k_y+p+\omega^2\right)\left(e^{\frac{2\sqrt{\zeta^2k_y+p+\omega^2}}{\sqrt{k_z}}}-1\right)\right)e^{\frac{(z_d+z_{\mathrm{dr}})\sqrt{\zeta^2k_y+p+\omega^2}}{\sqrt{k_z}}}
\end{aligned}
\tag{8}
$$

3 $\mathrm{h_b(z_d)}$ Solution

Equation (9) is $\mathrm{h_b(z_d)}$ numerator, equation (10) is $\mathrm{h_b(z_d)}$ denominator

$$
\begin{aligned}
&2\sqrt{\frac{2}{\pi}}p\omega h_{\mathrm{sD}}\sqrt{k_z}\left(\gamma p e^{\frac{(2z_d+z_{\mathrm{dr}}+2)\sqrt{\zeta^2 k_y+p+\omega^2}}{\sqrt{k_z}}}\right.\\
&+\left(\sqrt{k_z}\sqrt{\zeta^2 k_y+p+\omega^2}-\gamma p\right)e^{\frac{(z_d+z_{\mathrm{dr}})\sqrt{\zeta^2 k_y+p+\omega^2}}{\sqrt{k_z}}}\\
&-\left(\sqrt{k_z}\sqrt{\zeta^2 k_y+p+\omega^2}+\gamma p\right)e^{\frac{(z_d+z_{\mathrm{dr}}+2)\sqrt{\zeta^2 k_y+p+\omega^2}}{\sqrt{k_z}}}\\
&\left.+\gamma p e^{\frac{z_{\mathrm{dr}}\sqrt{\zeta^2 k_y+p+\omega^2}}{\sqrt{k_z}}}\right)-A\left(e^{\frac{2(z_d+1)\sqrt{\zeta^2 k_y+p+\omega^2}}{\sqrt{k_z}}}\right.\\
&\left.+1\right)\left(\gamma p\sqrt{\zeta^2 k_y+p+\omega^2}\left(1-e^{\frac{2z_{\mathrm{dr}}\sqrt{\zeta^2 k_y+p+\omega^2}}{\sqrt{k_z}}}\right)\right.\\
&\left.+\sqrt{k_z}\left(\zeta^2 k_y+p+\omega^2\right)\left(e^{\frac{2z_{\mathrm{dr}}\sqrt{\zeta^2 k_y+p+\omega^2}}{\sqrt{k_z}}}+1\right)\right)
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
&2p\sqrt{k_z}\sqrt{\zeta^2 k_y+p+\omega^2}\left(\gamma p\sqrt{\zeta^2 k_y+p+\omega^2}\left(e^{\frac{2\sqrt{\zeta^2 k_y+p+\omega^2}}{\sqrt{k_z}}}+1\right)\right.\\
&\left.+\sqrt{k_z}\left(\zeta^2 k_y+p+\omega^2\right)\left(e^{\frac{2\sqrt{\zeta^2 k_y+p+\omega^2}}{\sqrt{k_z}}}-1\right)\right)e^{\frac{(z_d+z_{\mathrm{dr}})\sqrt{\zeta^2 k_y+p+\omega^2}}{\sqrt{k_z}}}
\end{aligned}
\tag{10}
$$

4 Remarks

I am unsure where the mistake/s in the notes is/are but I am certain that these expressions satisfy the problem as stated under section 1. If I have misread the variables in translating them from original notes to mine, let me know and I can quickly fix it!