On Proving the Integral Definition of the Gamma Function for Non-Negative Real Numbers

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August 2019

One may be familiar with the fact that the gamma function of s, $\Gamma(s)$ for non-negative real numbers is defined by the integral:

$$\Gamma(s) = \int_0^\infty x^{s-1}\, e^{-x}\, dx$$

or simply

$$\Gamma(s) = \int_0^\infty \frac{x^{s-1}}{e^x}\, dx$$

But how can one prove this statement? There are quite a few ways to do so, but here is an example of one. One can try integrating by parts the integral before by setting $u = \frac{1}{e^x}$, and $dv = x^{s-1}$. From that one gets $du = -\frac{1}{e^x}$ with logarithmic differentiation by

$$y = e^{-x}$$

$$\ln y = -x \ln e$$

and since $\ln e = 1$,

$$\ln y = -x$$

differentiate both sides,

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(-x)$$

$$\frac{1}{y}\,\frac{dy}{dx} = -1$$

and multiply both sides by function y,

$$\frac{dy}{dx} = -e^{-x} = -\frac{1}{e^x} = du$$

To integrate x^{s-1}, apply the power rule for integration

$$\int x^n\, dx = \frac{x^{n+1}}{n+1}$$

and substitute $s+1$ for n to get

$$\int x^{s-1}\,dx = \frac{x^s}{s} = v$$

In result,

$$\int_0^\infty \frac{x^{s-1}}{e^x}\,dx = \left[\frac{x^s}{e^x s}\right]_{x=0}^\infty - \int_0^\infty -\frac{x^s}{e^x s}\,dx$$

or simply

$$\int_0^\infty \frac{x^{s-1}}{e^x}\,dx = \left[\frac{x^s}{e^x s}\right]_{x=0}^\infty + \int_0^\infty \frac{x^s}{e^x s}\,dx$$

The factor of $\frac{1}{s}$ can be pulled out from the integral to get

$$\int_0^\infty \frac{x^{s-1}}{e^x}\,dx = \left[\frac{x^s}{e^x s}\right]_{x=0}^\infty + \frac{1}{s}\int_0^\infty \frac{x^s}{e^x}\,dx$$

One can integrate by parts once more, by setting $u = \frac{1}{e^x}$, and $dv = x^s$. It has been proven earlier that $\frac{d}{dx}(\frac{1}{e^x}) = -\frac{1}{e^x} = du$ To integrate x^s, apply the same power rule for integration

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

and substitute s for n,

$$\int x^s\,dx = \frac{x^{s+1}}{s+1} = v$$

When substituted into the equation before, one has

$$\int_0^\infty \frac{x^{s-1}}{e^x}\,dx = \left[\frac{x^s}{e^x s} + \frac{x^{s+1}}{e^x s(s+1)}\right]_{x=0}^\infty - \int_0^\infty -\frac{x^{s+1}}{e^x s(s+1)}\,dx$$

or when simplified,

$$\int_0^\infty \frac{x^{s-1}}{e^x}\,dx = \left[\frac{x^s}{e^x s} + \frac{x^{s+1}}{e^x s(s+1)}\right]_{x=0}^\infty + \frac{1}{s(s+1)}\int_0^\infty \frac{x^{s+1}}{e^x}\,dx$$

There seems to be a pattern, resembling a sum, but before making final conclusions it is better to integrate by parts one more time by setting $u = \frac{1}{e^x}$, and $dv = x^s$. Again, it has been proven earlier that $\frac{d}{dx}(\frac{1}{e^x}) = -\frac{1}{e^x} = du$ To integrate x^{s+1}, apply the power rule for integration

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

and substitute $s+1$ for n,

$$\int x^{s+1}\,dx = \frac{x^{s+2}}{s+2} = v$$

Substituting into the previous equation and simplifying further, one gets

$$\int_0^\infty \frac{x^{s-1}}{e^x}\,dx = \left[\frac{x^s}{e^x s} + \frac{x^{s+1}}{e^x s(s+1)} + \frac{x^{s+2}}{e^x s(s+1)(s+2)}\right]_{x=0}^{\infty}$$
$$+\frac{1}{s(s+1)(s+2)}\int_0^\infty \frac{x^{s+2}}{e^x}\,dx$$

There are a couple details to be noted here, for instance the increasing integer value that is added to exponent s of x that correlates with the n^{th} term in the sum minus one. In the denominator, e^x is a common factor, but the rest can be expressed as a partial product that depends on the index quantity of the infinite sum. Putting everything together, one can express the gamma function $\Gamma(s)$ as

$$\Gamma(s) = \left[\sum_{n=0}^{\infty} \frac{x^{s+n}}{e^x \prod_{m=0}^{n}(s+m)}\right]_{x=0}^{\infty}$$

Since the infinite sum has external factors x^s and e^x, they can be pulled out of the sum, such that our equation looks like this:

$$\Gamma(s) = \left[\frac{x^s}{e^x}\sum_{n=0}^{\infty} \frac{x^{n}}{\prod_{m=0}^{n}(s+m)}\right]_{x=0}^{\infty}$$

The partial product in the denominator of the sum

$$\prod_{m=0}^{n}(s+m)$$

can be rewritten as

$$s\prod_{m=1}^{n}(s+m)$$

The Pochhamer rising factorial function $x^{(n)}$ can be defined as

$$x^{(n)} = \prod_{m=1}^{n}(x+m-1)$$

If x is substituted by $s+1$, one gets

$$(s+1)^{(n)} = \prod_{m=1}^{n}(s+m)$$

and so from there it can be derived that the term

$$s\prod_{m=1}^{n}(s+m)$$

can be written as

$$s(s+1)^{(n)}$$

Substituting into the original sum, $\Gamma(s)$ is expressed as

$$\Gamma(s) = \left[\frac{x^s}{e^x}\sum_{n=0}^{\infty}\frac{x^n}{s(s+1)^{(n)}}\right]_{x=0}^{\infty}$$

and when $\frac{1}{s}$ is factored out of the sum, it becomes

$$\Gamma(s) = \left[\frac{x^s}{se^x}\sum_{n=0}^{\infty}\frac{x^n}{(s+1)^{(n)}}\right]_{x=0}^{\infty}$$

The Pocchamer factorial has a property that defines it as

$$x^{(n)} = \frac{\Gamma(x+n)}{\Gamma(x)}$$

and if x is substituted by $s+1$, one gets

$$(s+1)^{(n)} = \frac{\Gamma(s+n+1)}{\Gamma(s+1)}$$

Plugging that in to our equation for $\Gamma(s)$, it becomes

$$\Gamma(s) = \left[\frac{x^s}{se^x}\sum_{n=0}^{\infty}\frac{x^n\Gamma(s+1)}{\Gamma(s+n+1)}\right]_{x=0}^{\infty}$$

and again the external factor of $\Gamma(s+1)$ can be pulled out of the infinite sum to get

$$\Gamma(s) = \left[\frac{x^s\Gamma(s+1)}{se^x}\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(s+n+1)}\right]_{x=0}^{\infty}$$

The function $\Gamma(s+1)$ is equal to $s\Gamma(s)$. This can be plugged in to our previous equation to have

$$\Gamma(s) = \left[\frac{x^s\Gamma(s)}{e^x}\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(s+n+1)}\right]_{x=0}^{\infty}$$

The expression

$$\frac{x^s\Gamma(s)}{e^x}\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(s+n+1)}$$

is the power series expansion for the lower incomplete gamma function $\gamma(s,x)$. And therefore the gamma function can be written as

$$\Gamma(s) = [\gamma(s,x)]_{x=0}^{\infty}$$

or

$$\Gamma(s) = \gamma(s, \infty) - \gamma(s, 0)$$

$\gamma(s, \infty)$ becomes the complete gamma function $\Gamma(s)$, while $\gamma(s, 0)$ breaks down to 0, and the equation becomes

$$\Gamma(s) = \Gamma(s) - 0$$

and finally

$$\Gamma(s) = \Gamma(s)$$

From this it can be said that the equation

$$\Gamma(s) = \int_0^\infty x^{s-1}\, e^{-x}\, dx$$

is true.