A Derivation of the Wave Equation from Maxwell's Equations

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What the Heck Is the Wave Equation

The Wave Equation Satisfies the Following PDE

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial x^2} \tag{1}$$

where psi is given by the general solution obtained via separation of variables.
Take note that this is the one dimensional form of the wave equation (one, unspecified component of a column matrix). The multidimentional wave equation satisfies the following PDE.

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \nabla^2 \circ \psi \tag{2}$$

The Derivation Itself

By Faraday's Law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3}$$

Arbitrarily take the curl of both sides of (3)

$$\nabla \times \nabla \times \mathbf{E} = -\nabla \times \frac{\partial \mathbf{B}}{\partial t} \tag{4}$$

Note the following vector identity

$$\nabla \times \nabla \times \mathbf{A} = \nabla\big(\nabla \circ \mathbf{A}\big) - \nabla^2 \mathbf{A}$$

Pay attention to the left side of (4) and apply this identity

$$\nabla\big(\nabla \circ \mathbf{E}\big) - \nabla^2 \mathbf{E} = -\nabla \times \frac{\partial \mathbf{B}}{\partial t}$$

Now take a look at the right side an note that the derivative operator commutes

$$\nabla\big(\nabla \circ \mathbf{E}\big) - \nabla^2 \mathbf{E} = -\frac{\partial}{\partial t}\big(\nabla \times \mathbf{B}\big) \tag{5}$$

The Derivation Itself Part II

Cool, now take note of the Ampere-Maxwell equation below

$$\nabla \times \mathbf{B} = \epsilon(\ \mathbf{J} + \mu \frac{\partial \mathbf{E}}{\partial t}) \tag{6}$$

Now apply this to (5)

$$\nabla(\nabla \circ \mathbf{E}) - \nabla^2 \mathbf{E} = \frac{\partial}{\partial t}(\ \mathbf{J} + \mu \frac{\partial \mathbf{E}}{\partial t}) \tag{7}$$

Assume that the current density, J, is equal to zero and apply this to (7)

$$\nabla(\nabla \circ \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Assume, also, that

$$\nabla \circ \mathbf{E} = 0$$

meaning that the divergence of the electric field has to necessarily be zero. Apply this assumption to (7) as well.

The Derivation Itself Part III

Well look at that! We have the wave equation!

$$\nabla^2 \mathbf{E} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

What is c?

$$\frac{1}{c^2} = \mu\epsilon$$

so

$$c = \sqrt{\frac{1}{\mu\epsilon}}$$

which is the propagation speed of light waves, confirmed by experiment.