Kelly bet sizing[1]

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[1] as discussed in p. 74 of Aaron Brown. *Red-Blooded Risk*. Wiley Finance, 2011

This document derives Kelly's rule for optimal bet sizing.

Here's the game. You have a $1000 stake. You face a series of 60 wins and 40 losses. Their order is chosen by me. You choose each bet's size. After you tell me the size, I get to spend, or not, one of my 40 kills—if I have any left. Your win is my loss, and vice-versa. After 100 rounds, we're done.

If you always bet the same fixed percentage of your stake, then[2] I have no strategic role to play in determining your overall takeaway. I can't affect your outcome if you play that way.

[2] since multiplication is commutative

The Math

I'll denote by a the aggressiveness of your bets. If you win a bet then $\texttt{stake}_{t+1} = \texttt{stake}_t \cdot (1+a)$ and if you lose a bet then $\texttt{stake}_{t+1} = \texttt{stake}_t \cdot (1-a)$. A win, then a loss, turns out the same as a loss, then a win:

$$\begin{aligned}\texttt{stake}_{t+2} &= (1+a)\cdot(1-a)\cdot\texttt{stake}_t \\ &= (1-a)\cdot(1+a)\cdot\texttt{stake}_t.\end{aligned}$$

In these terms, Kelly's question is: what is the optimal[3] aggressiveness a to **maximise your final takeaway**[4]?

[3] assuming, to remove my strategic input, that it's the same fraction a every time

[4] i.e., $\arg\max_{\{a\}} \texttt{stake}_{100}$

The Answer

> Twenty percent is the perfect amount to bet. Betting a higher or a lower fraction means doing worse. ... While there are people who dislike the Kelly criterion for various reasons, no intelligent person disputes this aspect of the result.
>
> —Brown, p. 76

http://books.google.com/books?id=eu925DG2xeYC&lpg=PP1&dq=red-blooded%20risk&pg=PT60#v=snippet&q=mathematical&f=false

Solving for optimal aggressiveness

This can be solved with calculus 101. Since the order doesn't matter and there's a fixed number of wins and losses, we can write the following formula for your total winnings at 60/40 odds:

$$\$1000 \cdot (1+a)^{60} \cdot (1-a)^{40} \tag{take}$$

. To find the optimal aggressiveness, set the derivative of equation (take), with respect to a, equal to zero.[5] The symbolic derivative of

$$(1+a)^{60} \cdot (1-a)^{40}$$

with respect to a is[6]:

$$60 \cdot (1+a)^{59} \cdot (1-a)^{40} \quad - \quad 40 \cdot (1+a)^{60} \cdot (1-a)^{39} \tag{D}$$

. Setting (D) = 0 tells me a property that is true of the optimal a^*. Moving things around, that property can be restated as:

$$60 \cdot \frac{(1-a^*)^{40}}{(1-a^*)^{39}} \quad = \quad 40 \cdot \frac{(1+a^*)^{60}}{(1+a^*)^{59}}$$

which reduces to the much nicer

$$60 \cdot (1-a^*) \quad = \quad 40 \cdot (1+a^*).$$

Solving then for the optimal aggressiveness a^*: [7]

$$60 - 60a^* = 40 + 40a^* \tag{1}$$
$$20 = 100a^* \tag{2}$$

which is what Brown gets: 20% = $\frac{1}{5}$ for the optimal a^* against these odds.

The "Ed Thorp takeaway" is that risk management does not mean taking no risks. You can't sit on your hands. Or, as Bill Gross says it, avoid "portfolio mush". When you have an edge (like a 60/40 edge) you need to exploit it. Betting less than 20% of your stake against 60/40 odds is[8] definitively suboptimal.

[5] (There are further conditions to make sure this works, which I'm leaving out.)

[6] The product rule says

$$D(f \cdot g) = D(f) \cdot g + f \cdot D(g).$$

[7] Thanks, Artemy!

[8] In this model, at least.

References

Aaron Brown. *Red-Blooded Risk*. Wiley Finance, 2011.