

*Kelly bet sizing*<sup>1</sup>  
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<sup>1</sup> as discussed in p. 74 of  
Aaron Brown. *Red-Blooded Risk*. Wiley  
Finance, 2011

This document derives Kelly's rule for optimal bet sizing.

HERE'S THE GAME. You have a \$10,000 stake. You face a series of 60 wins and 40 losses. Their order is chosen by me. You choose each bet's size. After you tell me the size, I get to spend, or not, one of my 40 kills—if I have any left. Your win is my loss, and vice-versa. After 100 rounds, we're done.

If you always bet the same fixed percentage of your stake, then<sup>2</sup> I have no strategic role to play in determining your overall takeaway. I can't affect your outcome if you play that way.

<sup>2</sup> since multiplication is commutative

*The Math*

I'll denote by  $a$  the aggressiveness of your bets. If you win a bet then  $\text{stake}_{t+1} = \text{stake}_t \cdot (1 + a)$  and if you lose a bet then  $\text{stake}_{t+1} = \text{stake}_t \cdot (1 - a)$ . A win, then a loss, turns out the same as a loss, then a win:

$$\begin{aligned}\text{stake}_{t+2} &= (1 + a) \cdot (1 - a) \cdot \text{stake}_t \\ &= (1 - a) \cdot (1 + a) \cdot \text{stake}_t.\end{aligned}$$

<sup>3</sup> assuming, to remove my strategic input, that it's the same fraction  $a$  every time

In these terms, Kelly's question is: what is the optimal<sup>3</sup> aggressiveness  $a^*$  to **maximise your final takeaway**<sup>4</sup>?

<sup>4</sup> i.e.,  $\arg \max_{\{a\}} \text{stake}_{100} \stackrel{\text{def}}{=} a^*$

*The Answer*

Twenty percent is the perfect amount to bet. Betting a higher or a lower fraction means doing worse. ... While there are people who dislike the Kelly criterion for various reasons, no intelligent person disputes this aspect of the result.

—Brown, p. 76

*Solving for optimal aggressiveness*

This can be solved with calculus 101. Since the order doesn't matter and there's a fixed number of wins and losses, we can write the following formula for your total winnings at 60/40 odds:

$$\$1000 \cdot (1 + a)^{60} \cdot (1 - a)^{40} \tag{take}$$

To find the optimal aggressiveness, set the derivative of equation (take), with respect to  $a$ , equal to zero.<sup>5</sup> The symbolic derivative of

$$(1 + a)^{60} \cdot (1 - a)^{40}$$

with respect to  $a$  is:

$$60 \cdot (1 + a)^{59} \cdot (1 - a)^{40} - 40 \cdot (1 + a)^{60} \cdot (1 - a)^{39} \tag{D}$$

. Setting (D) = 0 tells me a property that is true of the optimal  $a^*$ . Moving things around, that property can be restated as:

$$60 \cdot \frac{(1 - a^*)^{40}}{(1 - a^*)^{39}} = 40 \cdot \frac{(1 + a^*)^{60}}{(1 + a^*)^{59}}$$

which reduces to the much nicer

$$60 \cdot (1 - a^*) = 40 \cdot (1 + a^*).$$

Solving then for the optimal aggressiveness  $a^*$ :

$$60 - 60a^* = 40 + 40a^* \tag{1}$$

$$20 = 100a^* \tag{2}$$

which is what Brown gets: 20% =  $\frac{1}{5}$  for the optimal  $a^*$  against these odds.

*Brown's conclusion*

The "Ed Thorp takeaway" is that risk management does not mean taking no risks. You can't sit on your hands. Bill Gross says to "avoid portfolio mush", which is similar & related.

Contrast this to "lazy CAPM" style thinking. When you have an edge (like a 60/40 edge) you need to exploit it. Betting less than 20% of your stake against 60/40 odds is<sup>6</sup> suboptimal. Less aggressive betting does not move you along an optimal frontier; it moves you *off* the optimal frontier.

*References*

Aaron Brown. *Red-Blooded Risk*. Wiley Finance, 2011.

<sup>5</sup> (There are further conditions to make sure this works, which I'm leaving out.)

The product rule says  $D(f \cdot g) = D(f) \cdot g + f \cdot D(g)$ .

Thanks, Artemy!

<sup>6</sup> In this model.