Homework 1

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1 Problem 1

Find the best-fitting quadratic $y = ax^2 + bx + c$ to the data

x	-2	-1	1	2	3
y	2	1	-1	0	2

Graph the results

1.1 Solution

The following equation is a heuristic for error for the given quadratic and data:

$$e(a, b, c) = \sum_{n=1}^{5} \left(ax_n^2 + bx_n + c - y\right)^2$$

Naturally we want to minimize the error, so we take the partial derivatives relative to all of our variables, a, b, and c and set them all = 0.

$$\frac{\partial e}{\partial a} = 0 = \sum_{n=1}^{5} (ax_n^2 + bx_n + c - y) \cdot x_n^2 = 115a + 27b + 19c - 26$$

$$\frac{\partial e}{\partial b} = 0 = \sum_{n=1}^{5} (ax_n^2 + bx_n + c - y) \cdot x_n = 27a + 19b + 3c$$

$$\frac{\partial e}{\partial a} = 0 = \sum_{n=1}^{5} (ax_n^2 + bx_n + c - y) = 19a + 3b + 5c - 4$$

We now have 3 equations in 3 variables. Organizing this information into a matrix, we get the following system of equations in the form $\mathbf{A}\vec{x} = \vec{b}$.

$$\mathbf{A}\vec{x} = \begin{pmatrix} 115 & 27 & 19 \\ 27 & 19 & 3 \\ 19 & 3 & 5 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \vec{b} = \begin{pmatrix} 26 \\ 0 \\ 4 \end{pmatrix}$$

After inverting $\mathbf{A}$ we can change the equation to $\mathbf{A}^{-1}\mathbf{A}\vec{x} = \vec{x} = \mathbf{A}^{-1}\vec{b}$. ($\mathbf{A}$ is invertible; this can be verified by taking the $\det \mathbf{A}$.)

$$\mathbf{A}^{-1}\vec{b} = \begin{pmatrix} 279/616 \\ -339/616 \\ -13/22 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Therefore our best-fit quadratic is $\boxed{y = \frac{279}{616}x^2 + \frac{-339}{616}x + \frac{-13}{22}}$.

1.2 Graph

2 Problem 2

Find the best fit to $f(x) = x$ on $0 < x < 1$ of the form

$$\psi(x) = \sum_{n=1}^{N} c_n \sin n\pi x$$

2.1 Solution

The total variance of our function $\psi(x)$ can be determined by taking the integral

$$\int_0^1 (\psi(x) - f(x))^2 \,\mathrm{d}x = \int_0^1 \psi(x)^2 - 2\psi(x)f(x) + f(x)^2 \mathrm{d}x$$
$$= \int_0^1 \sum_{n=1}^{N} c_n \sin n\pi x \sum_{m=1}^{N} c_m \sin m\pi x - 2x \sum_{n=1}^{N} c_n \sin n\pi x + x^2 \mathrm{d}x$$

To simplify the task, we can separate the one integral into 3 separate ones, which can be solved individually and recombined later.

$$= \int_0^1 \sum_{n=1}^{N}\sum_{m=1}^{N} c_n c_m \sin n\pi x \sin m\pi x \mathrm{dx} - \int_0^1 2x \sum_{n=1}^{N} c_n \sin n\pi x \mathrm{d}x + \int_0^1 x^2 \mathrm{d}x$$

Since sums are just the addition of terms, over which we are integrating, we can do the same thing again, moving the integral within the summations while factoring out constant terms c_i.

$$= \sum_{n=1}^{N}\sum_{m=1}^{N} c_n c_m \int_0^1 \sin n\pi x \sin m\pi x \mathrm{dx} - 2\sum_{n=1}^{N} c_n \int_0^1 x \sin n\pi x \mathrm{d}x + \int_0^1 x^2 \mathrm{d}x$$

The first integral has two important cases, $n = m$ and $n \neq m$. Meanwhile, the second can be solved with integration by parts. The third is trivially solvable.

2.1.1 Solving the first integral

$$\sum_{n=1}^{N}\sum_{m=1}^{N} c_n c_m \int_0^1 \sin n\pi x \sin m\pi x \mathrm{dx}$$
$$= \sum_{n=1}^{N}\sum_{m=1, m\neq n}^{N} c_n c_m \int_0^1 \sin n\pi x \sin m\pi x \mathrm{dx} + \sum_{n=1}^{N} c_n^2 \int_0^1 \sin^2 n\pi x \mathrm{dx}$$

$$= \sum_{n=1}^{N} \sum_{m=1, m \neq n}^{N} \frac{c_n c_m}{2} \int_0^1 \cos(n\pi x - m\pi x) - \cos(n\pi x + m\pi x) \mathrm{dx} + \sum_{n=1}^{N} \frac{c_n^2}{2} \int_0^1 \cos(0) - \cos 2n\pi x \mathrm{dx}$$

$$= \sum_{n=1}^{N} \sum_{m=1, m \neq n}^{N} c_n c_m \frac{m \sin(n\pi x) \cos(m\pi x) - n \sin(m\pi x) \cos(n\pi x)}{(n^2 - m^2)\pi} + \sum_{n=1}^{N} \frac{c_n^2}{2} x - \frac{\sin 2n\pi x}{2n\pi} \bigg|_0^1$$

Evaluating at 0 eliminates every term, leaving behind

$$= \sum_{n=1}^{N} \sum_{m=1, m \neq n}^{N} c_n c_m \left(\frac{m \sin(n\pi) \cos(m\pi)}{(n^2 - m^2)\pi} - \frac{n \sin(m\pi) \cos(n\pi)}{(n^2 - m^2)\pi} \right) + \sum_{n=1}^{N} \frac{c_n^2}{2} - \frac{\sin 2n\pi}{2n\pi}$$

Furthermore, since n is an integer, $\sin 2n\pi = 0$ for all values of n, simplifying our summation. What is tricky is that the double-sum will have some odd, but predictable behavior. For every value of n, either the first term, or the second will be 0 making it oscillate between the two. Unfortunately m will exhibit the exact same behavior, making this sum difficult to rewrite.

Consider the following two cases. First when $m = a$, and $n = b$. Then the inside term becomes

$$ab \left(\frac{a \sin(b\pi) \cos(a\pi)}{(b^2 - a^2)\pi} - \frac{b \sin(a\pi) \cos(b\pi)}{(b^2 - a^2)\pi} \right)$$

And now consider the sister case, when $m = b$, and $n = a$:

$$ba \left(\frac{b \sin(a\pi) \cos(b\pi)}{(a^2 - b^2)\pi} - \frac{a \sin(b\pi) \cos(a\pi)}{(a^2 - b^2)\pi} \right) = ab \left(\frac{-b \sin(a\pi) \cos(b\pi)}{-(a^2 - b^2)\pi} - \frac{-a \sin(b\pi) \cos(a\pi)}{-(a^2 - b^2)\pi} \right)$$

These are both cases that will exist in the sum, so we will end up adding them together.

$$ab \left(\frac{a \sin(b\pi) \cos(a\pi)}{(b^2 - a^2)\pi} - \frac{b \sin(a\pi) \cos(b\pi)}{(b^2 - a^2)\pi} \right) + ab \left(\frac{-b \sin(a\pi) \cos(b\pi)}{(b^2 - a^2)\pi} + \frac{a \sin(b\pi) \cos(a\pi)}{(b^2 - a^2)\pi} \right)$$

What we see is that these two values are equal, and we can use this information to simplify our partial derivatives later.

2.1.2 Solving the second integral

$$-2 \sum_{n=1}^{N} c_n \int_0^1 x \sin n\pi x \mathrm{d}x$$

Let $u = x$, $\mathrm{d}u = \mathrm{d}x$, $\mathrm{d}v = \sin(n\pi x)\mathrm{d}x$, $v = \frac{-\cos(n\pi x)}{n\pi}$. Using integration by parts:

$$= -2 \sum_{n=1}^{N} c_n \left(x \frac{-\cos(n\pi x)}{n\pi} \bigg|_0^1 - \int_0^1 \frac{-\cos(n\pi x)}{n\pi} \mathrm{d}x \right)$$

$$= -2 \sum_{n=1}^{N} c_n \left(x \frac{-\cos(n\pi x)}{n\pi} + \frac{\sin(n\pi x)}{n^2\pi^2} \bigg|_0^1 \right) = -2 \sum_{n=1}^{N} c_n \left(\frac{-\cos(n\pi)}{n\pi} + \frac{\sin(n\pi)}{n^2\pi^2} \right)$$

2.1.3 Solving the thrid integral

$$\int_0^1 x^2 \mathrm{d}x = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$$

2.1.4 Back to the big picture

$$e(c_1, c_2, c_3, \cdots c_N) = \int_0^1 \left(\psi(x) - f(x)\right)^2 \mathrm{d}x =$$

$$\sum_{n=1}^{N} \sum_{m=1, m\neq n}^{N} c_n c_m \left(\frac{m\sin(n\pi)\cos(m\pi)}{(n^2-m^2)\pi} - \frac{n\sin(m\pi)\cos(n\pi)}{(n^2-m^2)\pi} \right) + \sum_{n=1}^{N} \left[\frac{c_n^2}{2} + 2c_n \left(\frac{\cos(n\pi)}{n\pi} - \frac{\sin(n\pi)}{n^2\pi^2} \right) \right] + \frac{1}{3}$$

We are aiming to minimize the error, so we need to take the partial derivatives of this function relative to c_i.

$$\frac{\partial e}{\partial c_i} = 2 \sum_{m=1, m\neq i}^{N} c_m \left(\frac{m\sin(i\pi)\cos(m\pi)}{(i^2-m^2)\pi} - \frac{i\sin(m\pi)\cos(i\pi)}{(i^2-m^2)\pi} \right) + c_i + 2\left(\frac{\cos(i\pi)}{i\pi} - \frac{\sin(i\pi)}{i^2\pi^2} \right) = 0$$

TODO:: THIS IS WHERE I AM STUCK!

3 Problem 3

Prove that the following are norms on $\mathbb{R}^n$.

- $||\mathbf{x}||_1 = \sum_{i=1}^{n} |x_i|$
- $||\mathbf{x}||_2 = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}$

3.1 Criteria of a norm

For a function f to be a norm, it must satisfy the following properties.

- $f(c\mathbf{x}) = |c|\, f(\mathbf{x}); \mathbf{x} \in \mathbb{R}^n, c \in \mathbb{R}$
- $f(\mathbf{x}+\mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y}); \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$
- If $f(\mathbf{x}) = 0$, then $\mathbf{x} = \mathbf{0}; \mathbf{x} \in \mathbb{R}^n$

3.2 Taxicab norm, $||\mathbf{x}||_1$

3.2.1 Proof of absolute homogeneity

$$||c\mathbf{x}||_1 = \sum_{i=1}^{n} |cx_i| = \sum_{i=1}^{n} |c|\,|x_i| = |c| \sum_{i=1}^{n} |x_i| = |c|\, ||\mathbf{x}||_1 \checkmark$$

3.2.2 Proof of triangle inequality

$$||\mathbf{x}+\mathbf{y}||_1 = \sum_{i=1}^{n} |x_i + y_i| \leq \sum_{i=1}^{n} |x_i| + |y_i| = \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i| = ||\mathbf{x}||_1 + ||\mathbf{y}||_1 \checkmark$$

3.2.3 Proof of the properties of the zero vector

We shall prove the third criterion via the use of the contrapositive.

$$\text{Assume } \mathbf{x} \neq \mathbf{0}, \mathbf{x} \in \mathbb{R}^n \Rightarrow \exists j \text{ such that } x_j \neq 0, j \in [1,n], j \in \mathbb{N}$$

$$||\mathbf{x}||_1 = \sum_{i=1}^{n} |x_i| \geq |x_j| > 0 \Rightarrow \text{ If } ||\mathbf{x}||_1 = 0, \text{ then } \mathbf{x} = \mathbf{0}\checkmark$$

3.2.4 Conclusion

Therefore, the function $\sum_{i=1}^{n} |x_i|$ is a norm on $\mathbb{R}^n$.

3.3 Euclidean norm, $||\mathbf{x}||_2$

3.3.1 Proof of homogeneity

$$||c\mathbf{x}||_2 = \left(\sum_{i=1}^{n}(cx_i)^2\right)^{1/2} = \left(\sum_{i=1}^{n} c^2 x_i^2\right)^{1/2} = \left(c^2 \sum_{i=1}^{n} x_i^2\right)^{1/2} = c\left(\sum_{i=1}^{n} x_i^2\right)^{1/2} = |c|\,||\mathbf{x}||_2 \checkmark$$

3.3.2 The Cauchy-Bunyakovsky-Schwarz Inequality for Sums

This theorem will be necessary to prove the triangle inequality holds for the Euclidean length within the vector space.

The theorem states the following:

$$\text{For each } \mathbf{x} = (x_1, x_2, x_3, \ldots, x_n)^t \text{ and } \mathbf{y} = (y_1, y_2, y_3, \ldots, y_n)^t \in \mathbb{R}^n,$$

$$\mathbf{x^t y} = \sum_{i=1}^{n} x_i y_i \leq \left\{\sum_{i=1}^{n} x_i^2\right\}^{1/2} \left\{\sum_{i=1}^{n} y_i^2\right\}^{1/2} = ||\mathbf{x}||_2 \cdot ||\mathbf{y}||_2 \quad \text{(Cauchy-Bunyakovsky-Schwarz)}$$

The actual proof of this theorem is reserved for the end of this document.

3.3.3 Proof of triangle inequality

$$||\mathbf{x}+\mathbf{y}||_2 = \left(\sum_{i=1}^{n}(x_i+y_i)^2\right)^{1/2} = \left(\sum_{i=1}^{n} x_i^2 + 2x_i y_i + y_i^2\right)^{1/2} \leq$$

$$\left(\sum_{i=1}^{n} x_i^2 + 2\left\{\sum_{i=1}^{n} x_i^2\right\}^{1/2}\left\{\sum_{i=1}^{n} y_i^2\right\}^{1/2} + \sum_{i=1}^{n} y_i^2\right)^{1/2} = \left[\left(\left\{\sum_{i=1}^{n} x_i^2\right\}^{1/2} + \left\{\sum_{i=1}^{n} y_i^2\right\}^{1/2}\right)^2\right]^{1/2} =$$

$$\left\{\sum_{i=1}^{n} x_i^2\right\}^{1/2} + \left\{\sum_{i=1}^{n} y_i^2\right\}^{1/2} = ||\mathbf{x}||_2 + ||\mathbf{y}||_2 \checkmark$$

3.3.4 Proof of the properties of the zero vector

We shall prove the third criterion via the use of the contrapositive... again.

$$\text{Assume } \mathbf{x} \neq \mathbf{0}, \mathbf{x} \in \mathbb{R}^n \Rightarrow \exists j \text{ such that } x_j \neq 0, j \in [1, n], j \in \mathbb{N}$$

$$||\mathbf{x}||_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \geq (x_j^2)^{1/2} = |x_j| > 0 \Rightarrow \text{ If } ||\mathbf{x}||_2 = 0, \text{then } \mathbf{x} = \mathbf{0} \checkmark$$

3.3.5 Conclusion

Therefore, the function $\left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$ is a norm on $\mathbb{R}^n$.

4 Problem 4

The amount N of a radioactive substance in a sample as a function of time satisfies the following radioactive decay equation:

$$N'(t) = -kN(t)$$

where $k > 0$ is called the decay constant and characterizes the radioactive material.

4.1 1.

If N_0 is the initial amount of the radioactive substance, what is the amount at time t?

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = -kN(t) \rightarrow \frac{\mathrm{d}N(t)}{N(t)} = -k\mathrm{d}t$$

$$\int \frac{\mathrm{d}N(t)}{N(t)} = \int -k\mathrm{d}t \rightarrow \ln N(t) = -kt + C$$

$$N(t) = Ce^{-kt}, N(0) = Ce^{-k\cdot 0} = C = N_0$$

$$\boxed{N(t) = N_0 e^{-kt}}$$

4.2 2.

The half life of a radioactive substance is the time τ at which half of the original amount of the radioactive substance has decayed. Show that the τ satisfies the following equation:

$$k\tau = \ln(2)$$

$$N(\tau) = N_0/2 = N_0 e^{-k\tau} \rightarrow \frac{1}{e^{k\tau}} = \frac{1}{2} \rightarrow \ln e^{k\tau} = k\tau = \ln 2$$

4.3 3.

Carbon-14 is a radioactive isotope of Carbon-12 with a half life of $\tau = 5730$ years. Carbon-14 is being constantly created in the atmosphere and is accumulated by living organisms. While the organism lives, the amount of Carbon-14 in the organism is held constant. The decay of Carbon-14 is compensated with new amounts when the organism breathes or eats. When the organism dies, the amount of Carbon-14 in its remains decays. So the balance between normal and radioactive carbon in the remains changes in time. If certain remains are found containing an amount of 14% of the original amount of Carbon-14, find the date of the remains.

$$k\tau = 5730 \text{ years} = \ln 2 \to k = 1.2097 \cdot 10^{-4}/\text{year}$$

$$N(t) = 0.14N_0 = N_0 e^{-1.2097\cdot 10^{-4}t} \to 1.2097 \cdot 10^{-4}t = \ln .14^{-1}$$

$$\boxed{t = 16,253 \text{ years}}$$

5 Problem 5

A tank has a salt mass $Q(t)$ dissolved in a volume $V(t)$ of water at a time t. Water is pouring into the tank at a rate $r_i(t)$ with a salt concentration $q_i(t)$. Water is also leaving the tank at a rate $r_o(t)$ with a salt concentration $q_o(t)$. Recall that a water rate r means water volume per unit time, and a salt concentration q means salt mass per unit volume. We assume that the salt entering in the tank gets instantaneously mixed. Q satisfies the following equation:

$$Q'(t) = a(t)Q(t) + b(t)$$

where

$$a(t) = -\frac{r_o}{(r_i - r_o)t + V_0},\ V(0) = V_0,\ b(t) = r_i q_i(t)$$

5.1 1.

Write down the [homogeneous] solution to $Q(t)$.

$$Q'(t) - a(t)Q(t) = b(t)$$

This is a First Order Linear Differential Equation. Our first step is to find $\mu(t)$ where

$$\mu(t) = e^{\int -a(t)\mathrm{d}t} = e^{\int \frac{r_o}{(r_i - r_o)t + V_0}\mathrm{d}t}$$

Multiplying both side of the equation by $\mu(t)$ allows us to treat the left hand side of the equation as the derivative of a product of function

$$\mu(t)\left[Q'(t) - a(t)Q(t)\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left[\mu(t)Q(t)\right] = \mu(t)b(t)$$

This works because of the cleverly chosen integrating factor $\mu(t)$ for which $\mu'(t) = \mu(t)(-a(t))$.

$$\int \frac{\mathrm{d}}{\mathrm{d}t}\left[\mu(t)Q(t)\right]\mathrm{d}t = \mu(t)Q(t) = \int \mu(t)b(t)\mathrm{d}t$$

Therefore

$$Q(t) = \frac{\int \mu(t)b(t)\mathrm{d}t}{\mu(t)} = \frac{\int e^{\int \frac{r_o}{(r_i - r_o)t + V_0}\mathrm{d}t} r_i q_i(t)\mathrm{d}t}{e^{\int \frac{r_o}{(r_i - r_o)t + V_0}\mathrm{d}t}}$$

Unfortunately, without knowing whether $r_i(t)$, $r_o(t)$, or $q_i(t)$ are independent of t, we cannot simplify this any further.

5.2 2.

Let $r_i = r_o = r$ and assume that fresh water is coming into the tank, hence $q_i = 0$. Then, find the time t_1, such that the salt concentration in the tank $Q(t)/V(t)$ is 1% the initial value. Write that time t_1 in terms of the rate r and initial water volume V_0.

$$Q(t) = \frac{C}{e^{\int \frac{r}{(r-r)t + V_0}\mathrm{d}t}} = \frac{C}{e^{\int \frac{r}{V_0}\mathrm{d}t}} = \frac{C_1}{e^{rt/V_0 + C_2}}$$

$$Q(0) = \frac{C_1}{e^{C_2}} = Q_0 \to Q(t) = Q_0 e^{-rt/V_0}, V(t) = V_0 + (r_i - r_o)t = V_0 + (r - r)t = V_0$$

$$\frac{Q(t_1)}{V(t_1)} = 1\% \frac{Q_0}{V_0} = \frac{Q_0 e^{-rt_1/V_0}}{V_0} \to 1\% = e^{-rt_1/V_0} \to \ln 100 = \frac{rt_1}{V_0} \to \boxed{t_1 = \frac{V_0 \ln 100}{r}}$$

6 Proof of The Cauchy-Bunyakovsky-Schwarz Inequality for Sums

This proof comes courtesy of Richard L. Burden and J. Douglas Faires, in the textbook Numerical Analysis, Ninth Ed.

If $\mathbf{y} = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$, the result is immediate because both sides of the inequality are zero.

Suppose $\mathbf{y} \neq \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$. Note that for each $\lambda \in \mathbb{R}$ we have

$$0 \leq ||\mathbf{x} - \lambda\mathbf{y}||_2^2 = \sum_{i=1}^{n}(x_i - \lambda y_i)^2 = \sum_{i=1}^{n} x_i^2 - 2\lambda \sum_{i=1}^{n} x_i y_i + \lambda^2 \sum_{i=1}^{n} y_i^2,$$

so that

$$2\lambda \sum_{i=1}^{n} x_i y_i \leq \sum_{i=1}^{n} x_i^2 + \lambda^2 \sum_{i=1}^{n} y_i^2 = ||\mathbf{x}||_2^2 + \lambda^2 ||\mathbf{y}||_2^2$$

However $||\mathbf{x}||_2 > 0$ and $||\mathbf{y}||_2 > 0$, so we can let $\lambda = ||\mathbf{x}||_2 / ||\mathbf{y}||_2$ to give us

$$\left(2\frac{||\mathbf{x}||_2}{||\mathbf{y}||_2}\right)\left(\sum_{i=1}^{n} x_i y_i\right) \leq ||\mathbf{x}||_2^2 + \frac{||\mathbf{x}||_2^2}{||\mathbf{y}||_2^2} ||\mathbf{y}||_2^2 = 2\,||\mathbf{x}||_2^2$$

Hence

$$\sum_{i=1}^{n} x_i y_i \leq ||\mathbf{x}||_2^2 \frac{||\mathbf{y}||_2}{||\mathbf{x}||_2} = ||\mathbf{x}||_2 \, ||\mathbf{y}||_2$$

QED.