Discrete math, 1 grade

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September 2019

1 Set theory

Definition 1.1.

set is set if and only if for every element we can say if it is in set or not.
$\in$ — *in*

1.1 Elementary properties

1.1.1 Russell's paradox

Definition 1.2.

$$\begin{aligned} &\textit{set } A \textit{ is good} \iff A \notin A. \\ &G = \{A \mid A \textit{ is good}\} \\ &G \in G? \end{aligned}$$

We can work with one big set, but what if we need more?

1.1.2 Axiom of power set

Definition 1.3.

$$\begin{aligned} &\forall \text{— } \textit{for all} \\ &\exists \text{— } \textit{exists} \\ &\exists! \text{— } \textit{exists and only one} \\ &A \subset B iff \forall x \in A\,(x \in B) \text{ — } A \textit{ is subset of } B \\ &|X| = \#X = \textit{number of elements in finite set} \end{aligned}$$

Definition 1.4. $\mathcal{P}(A) = 2^A = \mathcal{B}(A) = \{X \mid X \subset A\}$

Why do we write 2^A? Well, $\left|2^A\right| = 2^{|A|}$, so it's justified.

Axiom 1. $\exists$ *set* $\mathcal{A} \Rightarrow \exists$ *set* $2^{\mathcal{A}}$

1.1.3 Existence of product

Definition 1.5. $A \times B = \{(a,\ b) \mid a \in A \wedge b \in B\}$

Why do we write $A \times B$? Well, $|A \times B| = |A| \times |B|$, so it's justified.
$\exists$ *sets* $A, B \Rightarrow \exists\ A \times B$

1.2 Functions

Function is a relation between sets that associates to every element of a first set exactly one element of the second set.

Definition 1.6. $y = f(x)$ *— here f is a function and y is an image of x.*

Definition 1.7. *If f is function from A to B then A is domain of f and B is codomain of f.* $f\colon X \to Y$.

Definition 1.8. $\Gamma_f = \{(x, y) \in X \times Y \mid y = f(x)\}$ *— graph of a function.*

Definition 1.9.

$$\begin{aligned}
&if\ f\colon X \to Y:\\
&\forall x_1, x_2 \in X: x_1 \neq x_2\,(f(x_1) \neq f(x_2)) \iff f \text{ is an injection (one-to-one function)}\\
&\forall y \in Y\,(\exists x \in X\colon f(x) = y) \iff f \text{ is a surjection (function onto)}\\
&f \text{ is an injection and a surjection} \iff f \text{ is bijection (one-to-one correspondence)}
\end{aligned}$$

Definition 1.10. $Y^X = \{f \mid f\colon X \to Y\}$ *= set of functions from X to Y*
Why do we write Y^X*? Well,* $\left|Y^X\right| = |Y|^{|X|}$*, so it's justified.*

1.2.1 Composition

Definition 1.11. *if* $f\colon X \to Y$ *and* $g\colon Y \to Z$*, then* $g \circ f\colon X \to Z$ *and* $g \circ f(x) = g(f(x))$.
If $f\colon X \to Y,\ g\colon Y \to Z,\ h\colon Z \to W,$ *then* $h \circ (g \circ f) = (h \circ g) \circ f$.

Definition 1.12. *if X is a set, then* $Id_X = 1_X\colon X \to X, \forall x \in X\,(Id_x(x) = x)$ *— identity function.*

Definition 1.13. *if* $f\colon X \to Y,\ g\colon Y \to X,\ g \circ f = Id_X,\ f \circ g = Id_Y$*, then g is called inverse function to f or anti-function to f. We write* $g = f^{-1}$.

Theorem **(1).** For f exists inverse function $\iff$ f is a bijection.

Proof.

1. Proof that bijection is invertible

   If we have bijection $f\colon A \to B$ then $\forall b \in B\,(\exists! a \in A\colon f(a) = b)$. Let $g(b)$ be that a. Then $g = f^{-1}$.

2. Proof that invertible function is bijection.

(a) Proof that invertible function is injection.
Let's prove by contradiction. Let's say invertible function $f\colon A \to B$ isn't injection. That means $\exists a_1, a_2 \in A\colon a_1 \neq a_2 \wedge f(a_1) = f(a_2) = b$. Then $f^{-1}(b)$ is not defined. Contradiction.

(b) Proof that invertible function is surjection.
By yourself.

□

Theorem (2). $\forall \Omega \left(\exists\ bijection\ f\colon 2^{\Omega} \to \{0, 1\}^{\Omega}\right)$

Proof.

Definition 1.14.

$$\mathcal{X}_A(x) = \begin{cases} 1, \textit{if } x \in A \\ 0, \textit{if } x \notin A \end{cases}$$

$\mathcal{X}_A$ is called indicator function or characteristic function.
Lirical digression: $\mathcal{X}_{[0;+\infty)}$ is called Heaviside step fuction.

Definition 1.15. $N_\varphi = \left\{x \in \Omega \;\middle|\; \varphi(x) \neq 0\right\}$
N_φ is called support of the function φ
Obviously, $N_{\mathcal{X}_A} = A$ and $\mathcal{X}_{N_\varphi} = \varphi$. So $\mathcal{X}_A$ is a bijection on the set 2^Ω $\forall\Omega$.

□

1.2.2 What is this about? Who knows? Definitely not me

Definition 1.16. *Let $A,\ B \subset \Omega$. Then*

1. $A \cup B = \left\{x \;\middle|\; x \in A \vee x \in B\right\}$
2. $A \cap B = \left\{x \;\middle|\; x \in A \wedge x \in B\right\}$
3. $A \setminus B = \left\{x \in A \;\middle|\; x \notin B\right\}$
4. $\bar{A} = \Omega \setminus A$

Let's prove $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Proof.

1. Proof $\overline{A \cup B} \subset \overline{A} \cap \overline{B}$
$\forall x \in \overline{A \cup B} \left(x \notin A \cup B \Rightarrow x \notin A \wedge x \notin B \Rightarrow x \in \overline{A} \wedge x \in \overline{B} \Rightarrow x \in \overline{A} \cap \overline{B}\right)$
2. Proof $\overline{A} \cap \overline{B} \subset \overline{A \cup B}$
By yourself, dudes.

□

Let $A_1, \ldots,\ A_n \subset \Omega$. Let's choose random $x \in \Omega$ Let α_i be the answer on the question if $x \in A_i$. Then we have some function $f\colon \Omega \to \{0; 1\}^n$. So $\exists \Omega, A_1, \ldots, A_n\colon$ f is a surjection.