



# Aproximación Numérica de Ecuaciones Diferenciales Estocásticas

con Coeficientes Local Lipschitz

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6 de octubre de 2017

CONACYT-Universidad de Sonora

# Introducción

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# Por que EDEs?

En ocasiones

*EDO + ruido = Mejor modelo*

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## Crecimiento de Poblaciones

$$\frac{dN}{dt} = a(t)N(t) \quad N_0 = N(0) = \text{cte.}$$

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$$a(t) = r(t) + \text{"ruido"}$$

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## Circuitos Eléctricos

$$L \cdot Q''(t) + R \cdot Q'(t) + \frac{1}{C} \cdot Q(t) = F(t)$$

$$Q(0) = Q_0$$

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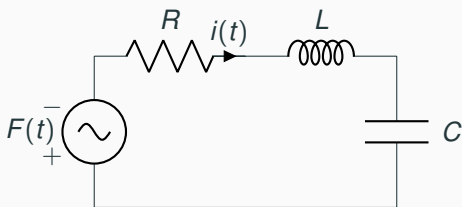
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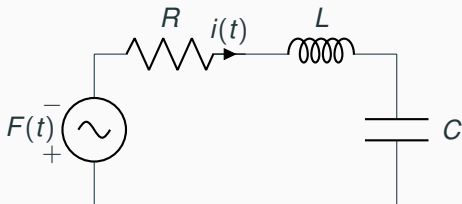
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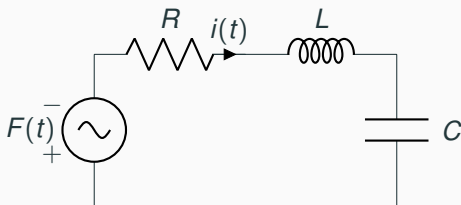
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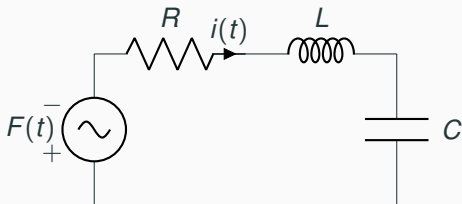
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Estima  $Z(t)$  observando  $Q(t)$

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Ejemplo

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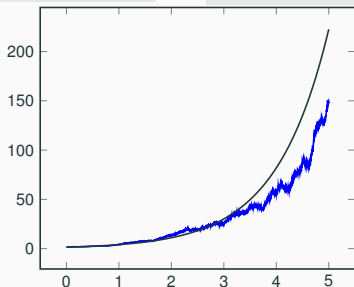
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**Solución analítica?**

muy RARA



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En ocasiones

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**Solución analítica?**

muy RARA

**Usa**

Teoría de diferencias finitas y  
haz una extensión estocástica.

## Objetivo de la charla

**Ilustrar** como aproximar soluciones de EDEs a partir de *conocimientos básicos* de los **métodos deterministas** y nociones muy elementales de variables aleatorias.

1. Introducción
2. Construcción de Métodos Numéricos
3. Aproximación Fuerte vs. Débil
4. EDEs con Coeficientes Local Lipschitz
5. Resumen y Comentarios Finales

# Construcción de Métodos Numéricos

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$$dx(t) = \underbrace{f(x(t), t)dt}_{\text{deriva}} + \underbrace{g(x(t), t)dB(t)}_{\text{difusión}},$$

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$$f: \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}^d, \quad g: \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}^{d \times m}$$

$$B(t) = (B_1(t), \dots, B_m(t))^T, \quad (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$$

EDE

$$x(t) = x_0 + \int_0^t f(x(s), s) ds + \int_0^t g(x(s), s) dB(s)$$

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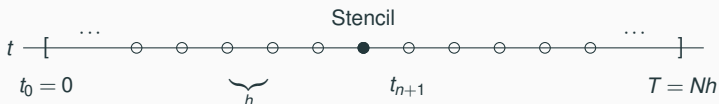
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# Idea general de la construcción

EDE

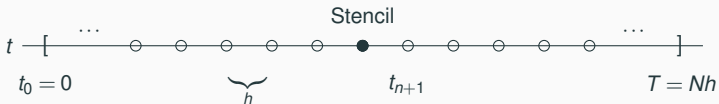
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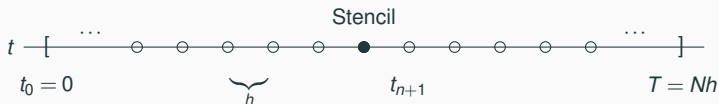


$$x(t_{n+1}) = x(t_n) + \int_{t_n}^{t_{n+1}} f(x(s), s) ds + \int_{t_n}^{t_{n+1}} g(x(s), s) dB(s)$$

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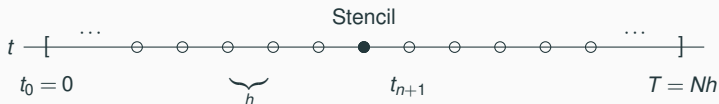


$$x(t_{n+1}) = x(t_n) + \underbrace{\int_{t_n}^{t_{n+1}} f(x(s), s) ds}_{\approx \text{det}} + \underbrace{\int_{t_n}^{t_{n+1}} g(x(s), s) dB(s)}_{\approx}$$

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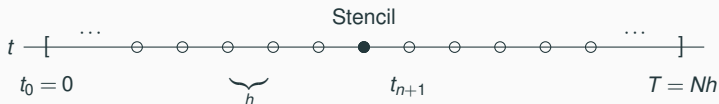


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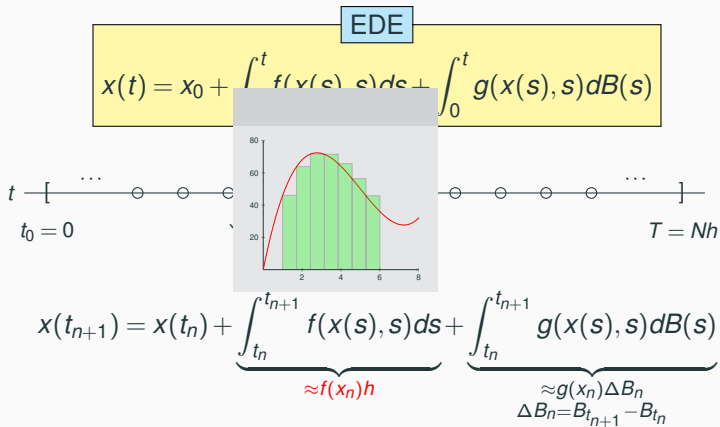
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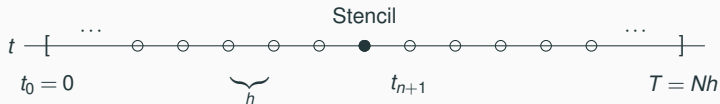
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$$X_0 = x_0, \quad X_n \approx x(t_n), \quad n = 1, \dots, N-1$$

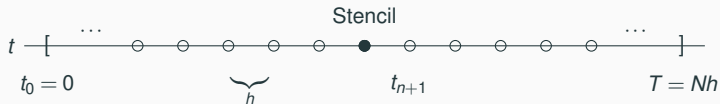
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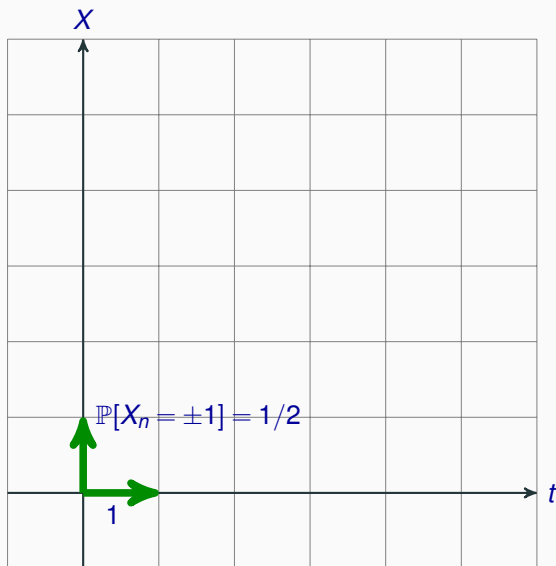
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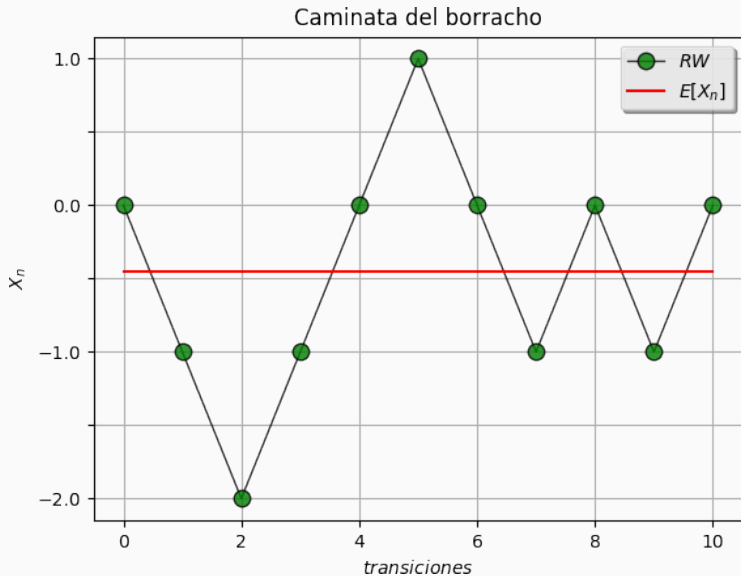
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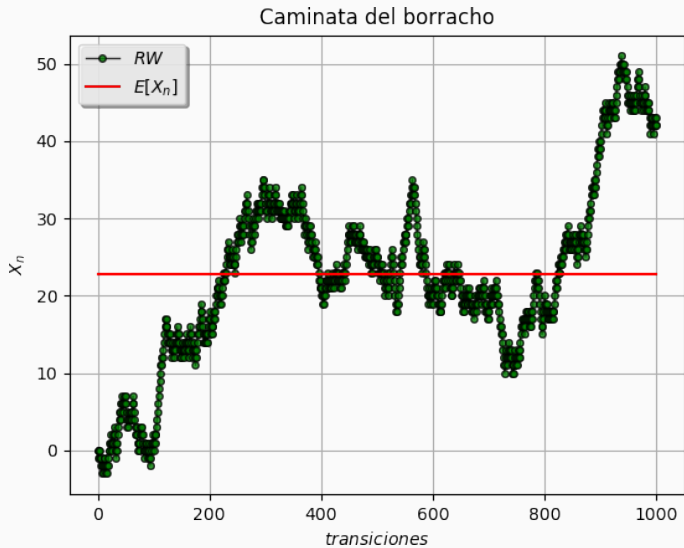


# Caminata del borracho

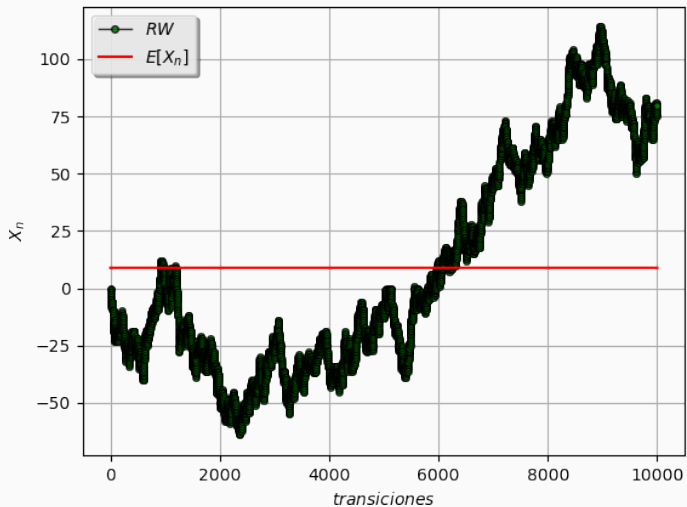


# Caminata Aleatoria

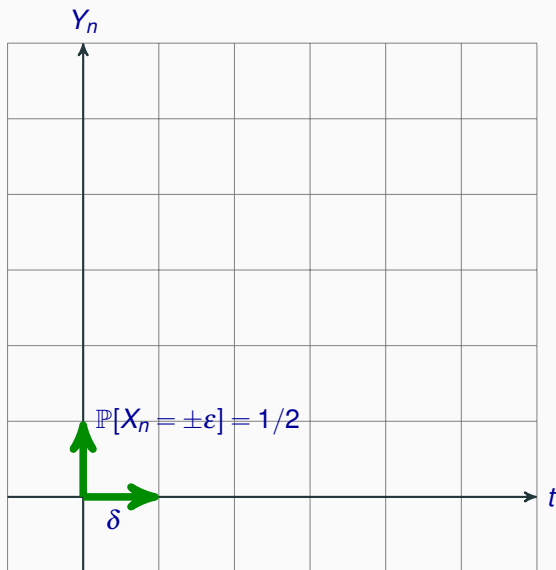




Caminata del borracho



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$\{X_n\}_{n=1}^{\infty}$  v.a..i.d

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Tomate  $\lambda \in \mathbb{R}$  fijo. Calcula

► característica  $\lim_{\delta,\varepsilon \rightarrow 0} \mathbb{E} \left[ e^{i\lambda Y_{\delta,\varepsilon}(t)} \right].$

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$$\lim_{\substack{\delta \rightarrow 0 \\ \varepsilon \rightarrow 0}} Y_{\delta,\varepsilon}$$

$$t = n\delta,$$

$$\begin{aligned} \mathbb{E} \left[ e^{i\lambda Y_{\delta,\varepsilon}(t)} \right] &= \prod_{j=1}^n \mathbb{E} \left[ e^{i\lambda X_j} \right] \\ &= \left( \mathbb{E} \left[ e^{i\lambda X_j} \right] \right)^n \\ &= \left( \frac{1}{2} e^{i\lambda \varepsilon} + \frac{1}{2} e^{-i\lambda \varepsilon} \right)^n \\ &= (\cos(\lambda h))^n \\ &= (\cos(\lambda h))^{\frac{t}{\delta}}. \end{aligned}$$

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$$\ln(u) = \frac{1}{\delta} \ln(\cos(\lambda\varepsilon))$$

Para  $x$  chirris!!!  $\ln(1+x) \approx x$

Para  $\varepsilon$  chirris!!!  $\cos(\lambda\varepsilon) \approx 1 - \frac{1}{2}\lambda^2\varepsilon^2.$

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$$t = n\delta, \quad u = (\cos(\lambda\varepsilon))^{\frac{1}{\delta}}$$

$$u \approx e^{-\frac{1}{2\delta} \lambda^2 \varepsilon^2}$$

$$\mathbb{E} \left[ e^{i\lambda Y_{\delta,\varepsilon}(t)} \right] \approx e^{-\frac{1}{2\delta} t \lambda^2 \varepsilon^2}.$$

$$\varepsilon^2 = \delta$$

$$\lim_{\delta \rightarrow 0} \mathbb{E} \left[ e^{i\lambda Y_{\delta,\sqrt{\delta}}(t)} \right] = e^{-\frac{1}{2} t \lambda^2}, \quad \lambda \in \mathbb{R}.$$

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$$\therefore B(t) \stackrel{\mathcal{D}}{=} \lim_{\delta \rightarrow 0} Y_{\delta, \sqrt{\delta}}(t)$$

## Teorema

Sea  $Y_{\delta,\varepsilon}(t)$  una caminata aleatoria que inicia en 0 de saltos  $\varepsilon$  y  $-\varepsilon$  con igual probabilidad en los tiempos  $\delta, 2\delta, 3\delta, \dots$ . Supongamos que  $\varepsilon^2 = \delta$ . Entonces para cada  $t \geq 0$ , el limite

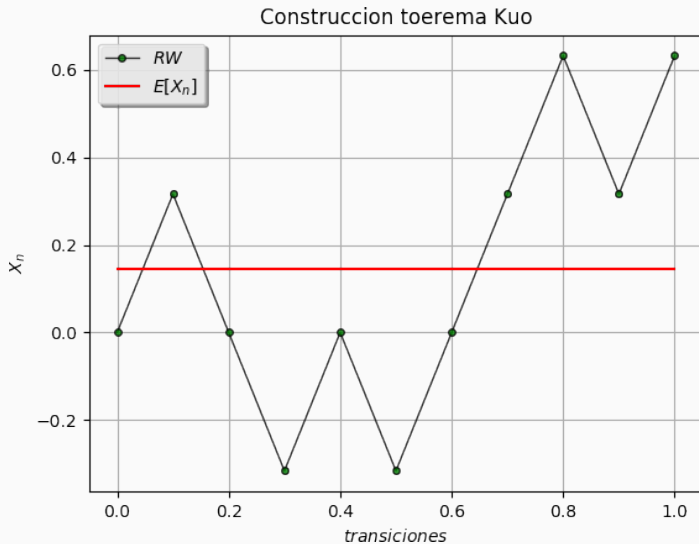
$$B(t) = \lim_{\delta \rightarrow 0} Y_{\delta, \sqrt{\delta}}(t),$$

existe en distribución. Además,

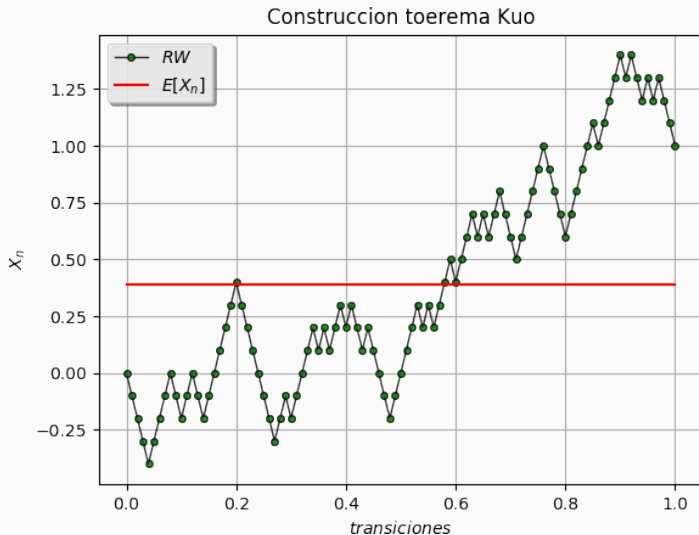
$$\mathbb{E} \left[ e^{i\lambda B(t)} \right] = e^{-\frac{1}{2}t\lambda^2}, \quad \lambda \in \mathbb{R}.$$

```
1 N = 10
2 T = 1.0
3 delta = T/np.float(N)
4 eps = 1.0/np.sqrt(np.float(N))
5 t = np.linspace(0,T,N+1)
6 b = np.random.binomial(1,.5, N) # bernulli 0,1
7 omega = 2.0 * b - 1           # bernulli -1,1
8 Xn = eps * (omega.cumsum())   # bernulli -h,h
9 Xn = np.concatenate(([0], Xn))
```

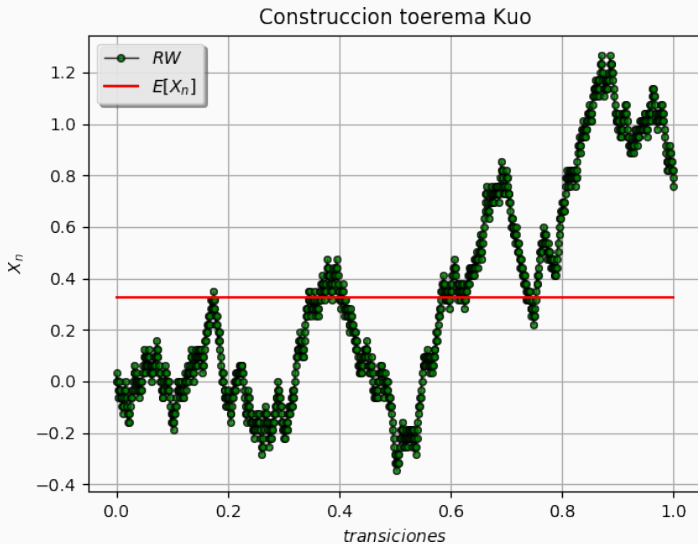
# Caminata Aleatoria de $n$ transiciones



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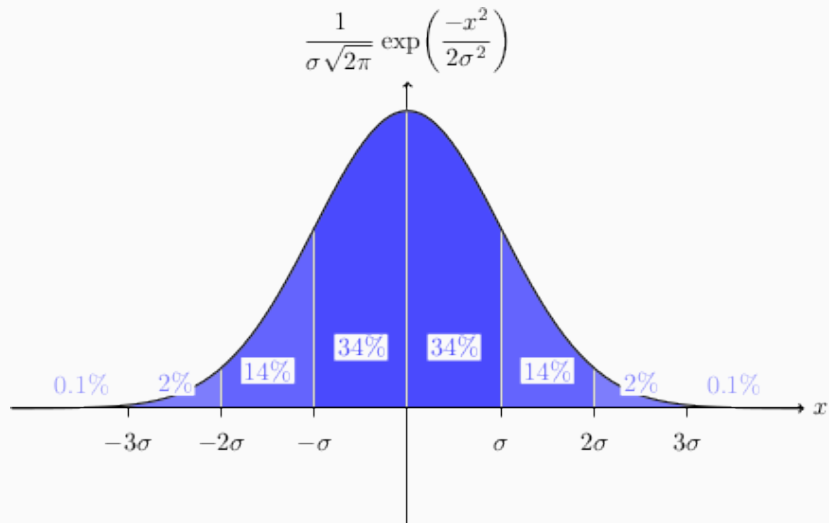
## Construcción

$$\varepsilon^2 = \delta$$

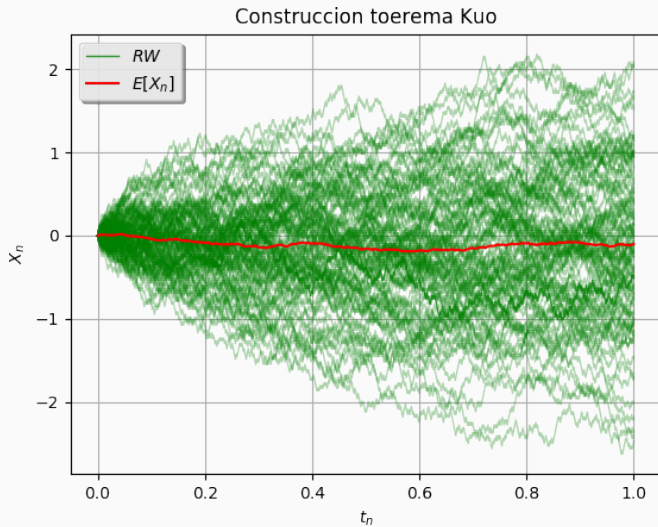
$$Y_{\delta,\varepsilon}(t) \xrightarrow[\delta,\varepsilon \rightarrow 0]{\mathcal{D}} B(t) \quad \forall t \geq 0$$

$$\mathbb{E} \left[ e^{i\lambda B(t)} \right] \xrightarrow[\delta,\varepsilon \rightarrow 0]{} e^{-\frac{1}{2}t\lambda^2}, \quad \lambda \in \mathbb{R}.$$

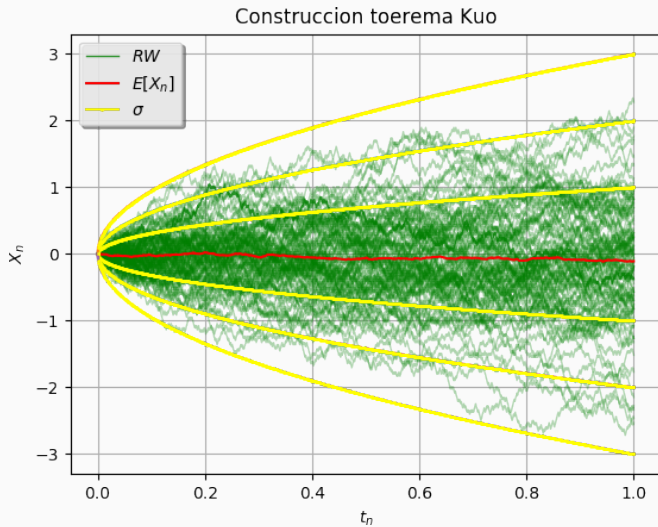
# Distribución Gaussiana



# Caminata Aleatoria en $[0, 1]$



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# Aproximación del MB en sentido Fuerte

## Definición

El movimiento Browniano  $B(t)$  es el único proceso que satisface:

- (I)  $B(0) = 0$  c.s.
- (II) Para  $0 \leq s \leq t$ ,  $B(t) - B(s) \sim \sqrt{t-s}N(0, 1)$ .
- (III) Para cualquier  $t_0 \leq t_1 \leq \dots \leq t_n \in [0, T]$ , las v.a  $B(t_i) - B(t_j)$  son independientes

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Entonces, dados  $t \in [0, T]$ , y un stencil

$$0 = t_0 \leq t_1 \leq \dots \leq t_{M-1} \leq t_M = t$$

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$$B(t_n) \approx \sum_{j=0}^n \Delta B_j, \quad \Delta B_0 := 0, \quad \Delta B_j \sim \sqrt{h}N(0, 1).$$

## **Aproximación Fuerte vs. Débil**

---

# Debil vs Fuerte

dada

$$dx(t) = f(x(t))dt + g(x(t))dB(t),$$

$$x(0) = x_0, \quad t \in [0, T]$$

## Debil

$$X_{n+1} = X_n + f(X_n)h + g(X_n) \underbrace{\Delta B_n}_{\approx \sqrt{h}\varepsilon_n}$$

$\mathbb{P}[\varepsilon_n = \pm 1] = 1/2$

## Fuerte

$$X_{n+1} = X_n + f(X_n)h + g(X_n) \underbrace{\Delta B_n}_{\approx \sqrt{h}\varepsilon_n}$$

$\varepsilon_n \sim N(0,1)$

# **EDEs con Coeficientes Local Lipschitz**

---

## Ejemplo:

$$dy(t) = -10 \operatorname{sgn}(y(t)) |y(t)|^{1.1} dt + 4 dW_t,$$

$$y_0 = 0, \quad t \in [0, 10]$$

$$\approx \mathbb{E}[|y(10)|], \quad 10^4 \text{ trayectorias,}$$

$$h = 10/N, \quad N = \{1, 2, \dots, 50\}$$

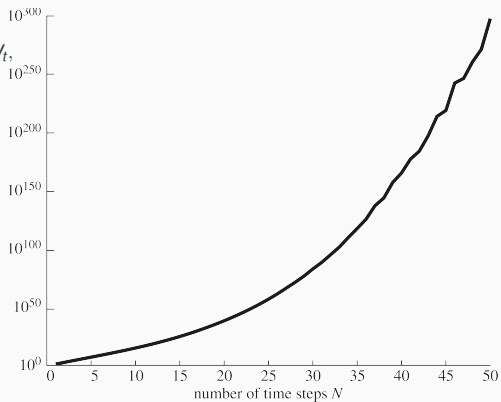
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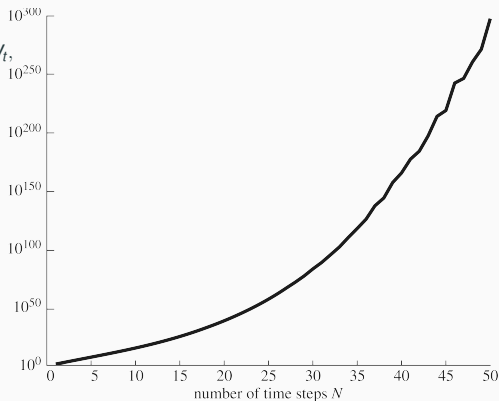
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Martin Hutzenthaler, Arnulf Jentzen, and Peter E. Kloeden.  
**Strong and weak divergence in finite time of euler's method  
for stochastic differential equations with non-globally  
lipschitz continuous coefficients.**

*Proceedings of the Royal Society of London A: Mathematical,  
Physical and Engineering Sciences*, 467(2130):1563–1576,  
2011.



# Modelos con Condiciones Local Lipschitz

## Lotka Volterra

$$dX_t = (\lambda X_t - kX_t Y_t)dt + \sigma X_t dW_t$$

$$dY_t = (kX_t Y_t - mY_t)dt$$

- Biología
- Finanzas
- Física
- Química

# Modelos con Condiciones Local Lipschitz

## Henston

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t \left( \sqrt{1 - \rho^2} dW_t^{(1)} + \rho dW_t^{(2)} \right)$$
$$dV_t = \kappa(\lambda - V_t) dt + \theta \sqrt{V_t} dW_t^{(2)}$$

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# Modelos con Condiciones Local Lipschitz

## Langevin

$$dX_t = -(\nabla U)(X_t)dt + \sqrt{2\varepsilon}dW_t$$

- Biología
- Finanzas
- Física
- Química

# Modelos con Condiciones Local Lipschitz

## Brusselator

$$dX_t = \left[ \delta - (\alpha + 1)X_t + Y_t X_t^2 \right] dt + g_1(X_t) dW_t^{(1)}$$

$$dY_t = \left[ \alpha X_t + Y_t X_t^2 \right] dt + g_2(X_t) dW_t^{(2)}$$

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## $\theta$ -Euler Maruyama

$$Y_{k+1} = Y_k + h(1 - \theta)f(Y_k) + \theta f(Y_{k+1}) + g(Y_k)\Delta W_k, \\ \theta \in [0, 1].$$

- **Implícitos:**

- $\theta$ -BEM

- FBEM

- **Explícitos:**

- Tamed EM

- Truncated

- Sabanis



Xuerong Mao and Lukasz Szpruch.

**Strong convergence and stability of implicit numerical methods for stochastic differential equations with non-globally lipschitz continuous coefficients.**

*Journal of Computational and Applied Mathematics*,  
238:14–28, January 2013.

## Forward-Backward Euler Maruyama

$$Y_k = Y_{k-1} + h(1 - \theta)f(Y_{k-1}) + \theta f(Y_k) + g(Y_{k-1})\Delta W_{k-1}$$
$$\widehat{Y}_{k+1} = \widehat{Y}_k + hf(Y_k) + g(Y_k)\Delta W_k, \quad \theta \in [0, 1].$$

- **Implícitos:**

- $\theta$ -BEM
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- **Explícitos:**

- Tamed EM
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## Tamed Euler Maruyama

$$Y_{k+1} = Y_k + \frac{hf(Y_k)}{1 + h\|f(Y_k)\|} + g(Y_k)\Delta W_k$$

- **Implícitos:**

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- **Explícitos:**

- Tamed EM
- Truncated
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Martin Hutzenthaler, Arnulf Jentzen, and Peter E. Kloeden.

**Strong convergence of an explicit numerical method for sdes with nonglobally lipschitz continuous coefficients.**

*The Annals of Applied Probability*, 22(4):1611–1641, August 2012.

## Truncated Euler Maruyama

- **Implicítos:**

- $\theta$ -BEM
- FBEM

- **Explícítos:**

- Tamed EM
- **Truncated**
- Sabanis

$$Y_{k+1} = Y_k + f_{\Delta}(Y_k)h + g_{\Delta}(Y_k)\Delta_k,$$
$$f_{\Delta}(x) := \left( |x| \wedge \mu^{-1}(h(\Delta)) \frac{x}{|x|} \right),$$
$$g_{\Delta}(x) := \left( |x| \wedge \mu^{-1}(h(\Delta)) \frac{x}{|x|} \right)$$



Xuerong Mao.

**The truncated euler-maruyama method for stochastic differential equations.**

*Journal of Computational and Applied Mathematics*,  
290:370 – 384, 2015.



## Euler Maruyama with varying coefficients

$$Y_{k+1} = Y_k + \frac{hf(Y_k) + g(Y_k)\Delta W_k}{1 + k^{-\alpha}(\|f(Y_k)\| + \|g(Y_k)\|)}, \quad \alpha \in (0, 1/2]$$

- **Implícitos:**

- $\theta$ -BEM
- FBEM

- **Explícitos:**

- Tamed EM
- Truncated
- Sabanis



Sotirios Sabanis.

**Euler approximations with varying coefficients:  
The case of superlinearly growing diffusion  
coefficients.**

*Ann. Appl. Probab.*, 26(4):2083–2105, 08 2016.

# Nuestra Propuesta

$$dy(t) = f(y(t))dt + g(y(t))dW_t, \quad f^{(j)}(x) = a_j(x)x^{(j)} + b_j(x^{(-j)})$$

$$f(y(t)) \approx \varphi_f(y(t_{\eta_+(t)}))$$

$$\eta(t) := k, \quad t \in [t_k, t_{k+1}), \quad k \geq 0,$$

$$\eta_+(t) := k+1, \quad t \in [t_k, t_{k+1}), \quad k \geq 0$$

$$\varphi_f(y(t_{\eta_+(t)})) = \frac{y(t_{\eta_+(t)}) - y(t_{\eta(t)})}{\int_{y(t_{\eta(t)})}^{y(t_{\eta_+(t)})} \frac{du}{a(y(t_{\eta(t)}))u+b}}$$

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**Hipótesis:**  $\forall x \in \mathbb{R}^d$

(A-1)  $\exists L_a, a_j(x) \leq L_a$

(A-2)  $|b_j(x^{(-j)})|^2 \leq L_b(1 + |x|^2)$

(A-3) Condiciones ceros de  $a_j(\cdot)$

## Teorema

Sea  $u \in \mathbb{R}^d$

$$v = u + h\varphi_f(v),$$

$$v = A^{(1)}(h, u)u + A^{(2)}(h, u)b(u).$$

► Def

$$v = A^{(1)}(h, u)u + A^{(2)}(h, u)b(u)$$

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## Método Explícito

$$Y_k^* = A^{(1)}(h, Y_k)Y_k + A^{(2)}(h, Y_k)b(Y_k),$$

$$Y_{k+1} = Y_k^* + g(Y_k^*)\Delta W_k,$$

**Hipótesis:**  $\forall x \in \mathbb{R}^d$

(A-1)  $\exists L_a, a_j(x) \leq L_a$

(A-2)  $|b_j(x^{(-j)})|^2 \leq L_b(1 + |x|^2)$

(A-3) Condiciones ceros de  $a_j(\cdot)$

## Contra ejemplo para los tamed

$$dy_1(t) = (\lambda - \delta y_1(t) - (1 - \gamma)\beta y_1(t)y_3(t)) dt - \sigma_1 y_1(t) dW_t^{(1)},$$

$$dy_2(t) = ((1 - \gamma)\beta y_1(t)y_3(t) - \alpha y_2(t)) dt - \sigma_1 y_2(t) dW_t^{(1)},$$

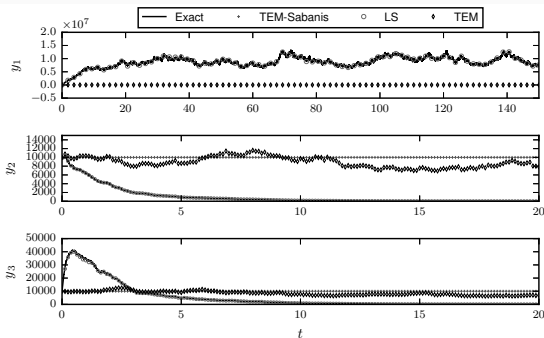
$$dy_3(t) = ((1 - \eta)N_0 \alpha y_2(t) - \mu y_3(t) - (1 - \gamma)\beta y_1(t)y_3(t)) dt - \sigma_2 y_3(t) dW_t^{(2)}$$

# Contra ejemplo para los tamed

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$\gamma = 0.5, \eta = 0.5, \lambda = 10^6,$   
 $\delta = 0.1, \beta = 10^{-8}, \alpha = 0.5,$   
 $N_0 = 100, \mu = 5, \sigma_1 = 0.1,$   
 $\sigma_2 = 0.1,$   
 $y_0 =$   
 $(10000, 10000, 10000.)^T,$   
 $h = 0.125.$

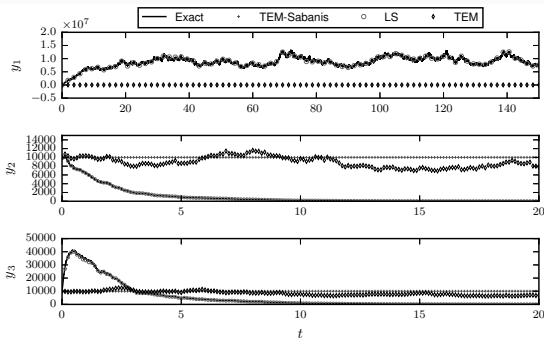
Exacta: BEM  $h = 10^{-5}$

# Contra ejemplo para los tamed

$$dy_1(t) = (\lambda - \delta y_1(t) - (1 - \gamma)\beta y_1(t)y_3(t)) dt - \sigma_1 y_1(t) dW_t^{(1)},$$

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# **Resumen y Comentarios Finales**

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# Función característica

## Definición (Función característica)

Sea  $X$  v. a., entonces,

$$\phi_X(t) = \mathbb{E} \left[ e^{itX} \right], \quad t \in \mathbb{R},$$

es la función característica de  $X$ .

## Teorema de continuidad

Sea  $\{X_n\}_{n=1}^{\infty}$  v.a., entonces

$$X_n \xrightarrow{\mathcal{D}} X \Leftrightarrow \phi_{X_n}(t) \rightarrow \phi_X(t).$$

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$$\int_0^T f(\cdot) d(\cdot)$$

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$$f : [0, T] \rightarrow \mathbb{R}$$

Determinista:

$$\int_0^T f(\cdot) dg(\cdot) \approx \sum_{j=0}^{N-1} f(t_j)(t_{j+1} - t_j)$$

# Integral Estocástica

Integral

$$\int_0^T f(\cdot) dB(\cdot)$$

$$f : [0, T] \times \Omega \rightarrow \mathbb{R}$$

Determinista:

$$\int_0^T f(\cdot) dg(\cdot) \approx \sum_{j=0}^{N-1} f(t_j)(t_{j+1} - t_j)$$

**Itô**

$$\approx \sum_{j=0}^{N-1} f(t_j)(B_{t_{j+1}} - B_{t_j})$$

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**Itô**

$$\approx \sum_{j=0}^{N-1} f(t_j)(B_{t_{j+1}} - B_{t_j})$$

**Stratonovich**

$$\approx \sum_{j=0}^{N-1} f\left(\frac{t_j + t_{j+1}}{2}\right)(B_{t_{j+1}} - B_{t_j})$$



# Existencia y unicidad de soluciones fuertes para EDEs

Sea  $dX_t = f(t, X_t)dt + g(t, X_t)dB_t$  en el sentido de Itô, t.q.

(EU1) (Medibles):  $f, g$  son  $\mathcal{L}^2$ -medibles en  $(t, x) \in [t_0, T] \times \mathbb{R}$ .

(EU2) (Lipschitz):  $\exists K > 0$  t.q.  $\forall t \in [t_0, T], x, y \in \mathbb{R}$ .

$$|f(t, x) - f(t, y)| \leq K|x - y|,$$

$$|g(t, x) - g(t, y)| \leq K|x - y|$$

(EU3) (De crecimiento lineal acotado):  $\exists K > 0$ , t.q.

$$\forall t \in [t_0, T], x \in \mathbb{R}$$

$$|f(t, x)|^2 \leq K^2(1 + |x|^2),$$

$$|g(t, x)|^2 \leq K^2(1 + |x|^2)$$

(EU4) (Condición inicial):  $X_{t_0}$  es  $\mathcal{F}_{t_0}$ -medible con  $\mathbb{E}[|X_{t_0}|] < \infty$ .

Entonces,  $\exists! X_t$  en  $[t_0, T]$  con  $\sup_{t_0 \leq t \leq T} \mathbb{E}(|X_t|^2) < \infty$ .

# Lema de Gronwall

## Lema (de Gronwall)

Sean  $\alpha, \beta : [t_0, T] \rightarrow \mathbb{R}$  funciones integrables t.q.

$$0 \leq \alpha(t) \leq \beta(t) + L \int_{t_0}^t \alpha(s) ds \quad t \in [t_0, T].$$

Entonces

$$\alpha(t) \leq \beta(t) + L \int_{t_0}^t e^{L(t-s)} \beta(s) ds$$

◀ Prueba

◀ Idea

# Desigualdad de Lyapunov

Sea  $X$  una v.a integrable y  $0 < q \leq p$  entonces

**Sea  $X$  una v.a integrable y  $0 < q \leq p$  entonces**

$$\mathbb{E}(|X|^q) \leq \mathbb{E}(|X|^p)^{\frac{q}{p}}$$

◀ Prueba

## Propiedades Integral de Itô

$$1. \mathbb{E} \left[ \int_0^T g(r) dB_r \right] = 0$$

$$2. \text{(Isometría)} \mathbb{E} \left[ \left( \int_0^T g(r) dB_r \right)^2 \right] = \int_0^T g^2(r) dr$$

## Apéndice A

$$A^{(1)}(h, u) := \begin{pmatrix} e^{ha_1(u)} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & e^{ha_d(u)} \end{pmatrix},$$

$$A^{(2)}(h, u) := \begin{pmatrix} \left(\frac{e^{ha_1(u)} - 1}{a_1(u)}\right) \mathbf{1}_{\{E_1^c\}} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \left(\frac{e^{ha_d(u)} - 1}{a_d(u)}\right) \mathbf{1}_{\{E_d^c\}} \end{pmatrix} + h \begin{pmatrix} \mathbf{1}_{\{E_1\}} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{1}_{\{E_d\}} \end{pmatrix},$$

$$E_j := \{x \in \mathbb{R}^d : a_j(x) = 0\}, \quad b(u) := (b_1(u^{(-1)}), \dots, b_d(u^{(-d)}))^T.$$

## Apéndice B: Resultado para ceros aislados

### Definición (DD respecto a $p$ )

$u, \mathbf{p} \in \mathbb{R}^2$ ,  $\alpha$  ángulo positivo respecto a eje- $x$  segmento  $\overline{u\mathbf{p}}$ .

$$f_\alpha(u) = \frac{\langle q - u, \nabla f(u) \rangle}{|u - q|}$$

derivada direccional respecto  $\mathbf{p}$  en  $u$ .

### Definición (Star-like set)

$S \subset \mathbb{R}^2$  es *star-like* respecto  $\mathbf{p}$ ,  $\forall s \in S$  el segmento abierto  $\overline{sp}$  esta en  $S$ .

### Teorema

- $\mathbf{p} \in \mathbb{R}^2$ ,  $S \subset \mathbb{R}^2$  *star-like* respecto  $\mathbf{p}$  en el dominio de  $f, g$ .
- En  $S$ ,  $f, g$  diferenciables,  $g_\alpha(s) \neq 0$ ,
- $f(\mathbf{p}) = g(\mathbf{p}) = 0$ ,  $\lim_{x \rightarrow \mathbf{p}} \frac{f_\alpha(x)}{g_\alpha(x)} = L$ ,

Entonces  $\lim_{x \rightarrow \mathbf{p}} \frac{f(x)}{g(x)} = L$ .



Al Fine and S Kass.

### Indeterminate forms for multi-place functions.

*Annales Polonici Mathematici*,  
18(1):59–64, 0 1966.

## Apéndice B: Condiciones para ceros de $a_j(\cdot)$

$E_j := \{x \in \mathbb{R}^d : a_j(x) = 0\}$  satisface alguno de los puntos:

(I)  $p \in E_j$  es un cero no aislado de  $a_j(\cdot)$  y:

- $D := \{u : e^{ha_j(u)} - 1 = a_j(u) = 0\}$ , es una curva suave que pasa por  $p$ .
- El vector canónico  $e_j$  es no tangente a  $D$ .
- Para cada  $p \in E_j$ , existe una bola  $B_r(p)$  t.q.

$$a_j \neq 0, \quad \frac{\partial a_j(u)}{\partial u(i)} \neq 0, \quad \forall u \in D \setminus B_r(p).$$

(II)  $p \in E_j$  es un cero aislado de  $a_j(\cdot)$  y:

- Para cada  $q \in E_j$ ,  $p$  no es punto límite de  $E_\alpha := \{x \in \mathbb{R}^d : (a_j)_\alpha(x) = 0\}$ .
- Para cada  $p \in E_j$  existe  $B_r(p)$ , t.q. la derivada direccional respecto a  $p$  satisface

$$(a_j)_\alpha(x) \neq 0, \quad \forall x \in B_r(p).$$