# Analysis of "Chaos in a Periodically Forced Predator-Prey Ecosystem Model" by Gary C. W. Sabin and Danny Summers

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## Objectives

In "Chaos in a Periodically Forced Predator-Prey Ecosystem Model", G. Sabin and D. Summers analyze the chaos that arises in a periodically forced system. They do this by adding a periodic forcing term to the simple predator-prey model introduced by Volterra. They then perform numerical experiments with varied parameter values and calculate Lyapunov exponents and fractal dimensions that suggest the system is chaotic. I attempt to reproduce some of their results and further analyze the system's chaos by varying initial conditions.

## Tools Used to Analyze Chaos

Below, are brief descriptions of the tools Sabin and Summers use to analyze the system's chaos.

- **Poincaré maps** show a "sequence of points generated by the intersection of a trajectory of a continuous dynamical system with a given surface in the phase space" (p. 96).
- The **Lyapunov exponent**, $\lambda > 0$, is found by calculating

$$\Delta(t) = \Delta(0)2^{\lambda t}$$

  where $\Delta(0)$ measures the small initial separation of the trajectories and $\Delta(t)$ measures separation at time $t$.
- The **fractal dimension** of a strange attractor measures "the extent to which trajectories on the attractor fill a region of phase space.
  - Sabin and Summers use the **Lyapunov dimension** $D_L$ which is defined by

$$D_L = 2 + \frac{\lambda_1}{\lambda_3}$$

    in three-dimensional phase space.
  - They also use the **correlation dimension** $D_C$ which relates to the spatial correlation of points on an attractor and is defined by

$$D_C = \lim_{r \to 0} \frac{\log C(r)}{\log r}.$$

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## Periodically Forced Model

Adding a periodic forcing term may be approached in many different ways. Sabin and Summers choose to validate the periodic variation as an intrinsic growth rate of the prey. So $\rho_* = \rho_0 + \rho_1(1 - \cos(\Omega t))$. After non-dimensionalization the system is now:

$$\frac{dX_1}{dt} = X_1(1 - X_1) - aX_1X_2 + \Gamma(1 - \cos(\omega t))X_1$$

$$\frac{dX_2}{dt} = -bX_2 + aX_1X_2$$

where $\Gamma = \frac{\rho_1}{\rho_0}$ and $\frac{\omega}{\rho_0}$.

*Note*: this system is analogous to forced nonlinear oscillators in physics such as the Duffing oscillator. This problem is essentially six dimensional because it is dependent on four "control" parameters ($a$, $b$, $\Gamma$, $\omega$) and two initial conditions ($X_1(0)$ and $X_2(0)$).

## Varying Control Parameter $b$

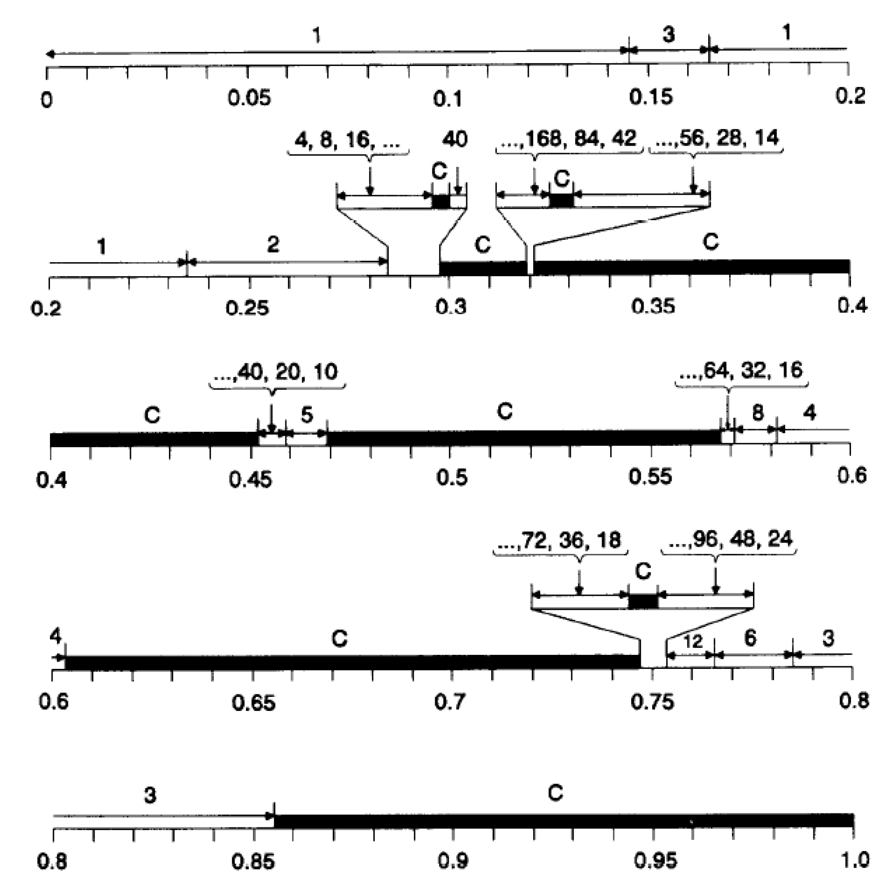

Figure 1: Dynamical behavior of the system with $a = 1$, $\Gamma = 8.7$, $\omega = 4$, $X_1(0) = 0.5$ and $X_2(0) = 1$. Chaotic solutions are denoted by $C$ and period-$n$ solutions by $n$. Obtained from [1].

## Varying Control Parameter $\Gamma$

In the following plots: $a = 1$, $b = 0.3$, $\omega = 4$, $X_1(0) = 0.5$ and $X_2(0) = 1$.

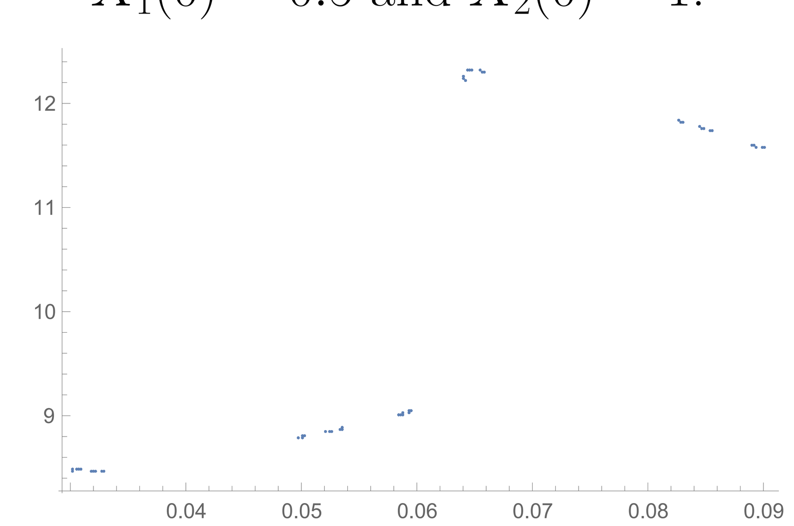

Figure 2: $\Gamma$ value: 8.6562.

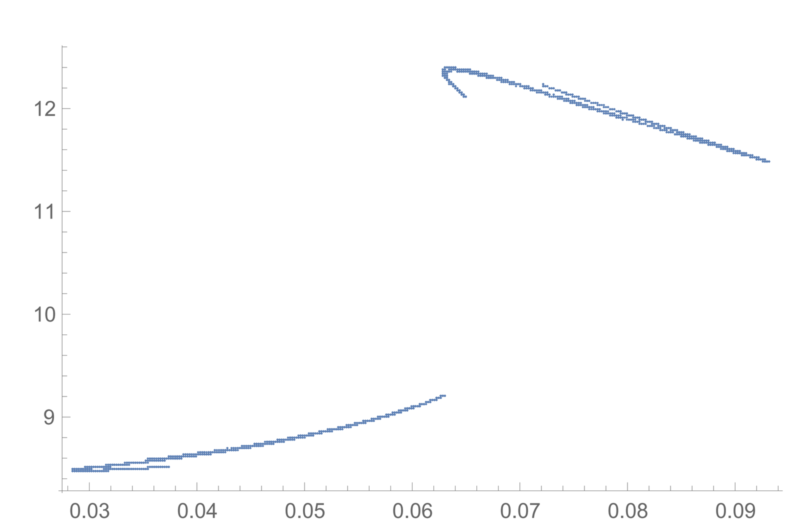

Figure 3: $\Gamma$ value: 8.68.

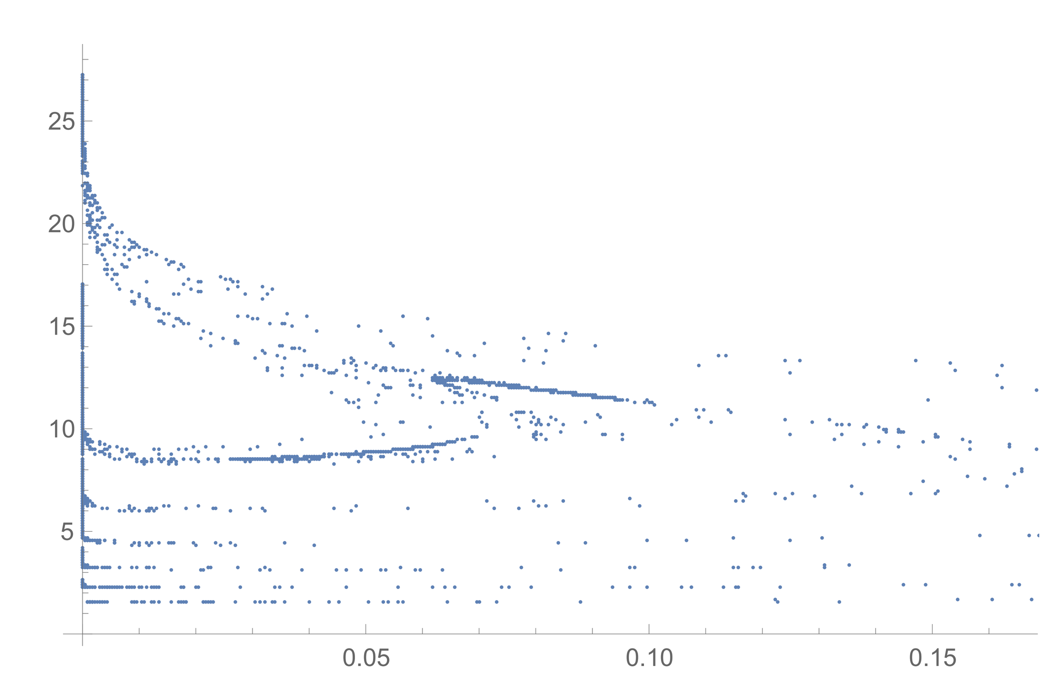

Figure 4: $\Gamma$ value: 8.7.

| $\Gamma$ | $\lambda_1$ | $\lambda_3$ | $D_L$ | $D_C1$ |
|---|---|---|---|---|
| 8.67 | 0.088 | -0.52 | 2.17 | 2.17 ± 0.01 |
| 8.69 | 0.17 | -0.60 | 2.28 | 2.26 ± 0.02 |
| 8.70 | 0.28 | -0.71 | 2.39 | 2.20 ± 0.01 |
| 8.90 | 0.32 | -0.75 | 2.43 | 2.06 ± 0.003 |
| 9.00 | 0.31 | -0.75 | 2.41 | 2.13 ± 0.004 |

Table 1: Comparison of Lyapunov Dimension and Correlation Dimension for strange attractors with $a = 1$, $b = 0.3$ and $\omega = 4$.

## Varying Initial Conditions

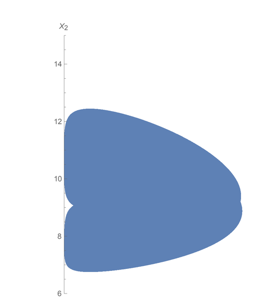

Figure 5: $\Gamma = 8.6$, $X_1(0) = 0.5$ and $X_2(0) = 1$.

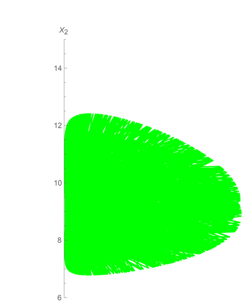

Figure 6: $\Gamma = 8.6$, $X_1(0) = 0.51$ and $X_2(0) = 1$.

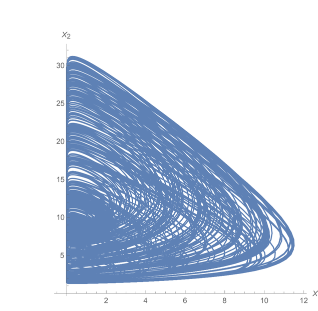

Figure 7: $\Gamma = 8.7$, $X_1(0) = 0.5$ and $X_2(0) = 1$.

## Conclusions

Sabin and Summers decided to analyze the system by varying the control parameters rather than the initial conditions because of computation time. They do this by employing a fourth order Runge-Kutta method with time step $10^{-2}$ and double-precision arithmetic. When varying $\Gamma$ values between $7 < \Gamma < 9$, period-doubling bifurcations appeared. This again is analogous to the logistic map or double pendulum bifurcation plot and tells us that chaos is present in the system. Varying the control parameter $b$ showed us that in a small interval $0 < b < 1$, many different types of solutions are present. In addition to reproducing their results, I varied initial conditions while keeping all other parameters fixed. Parametric plots show us that the solutions are very different.

## References

[1] Gary CW Sabin and Danny Summers.
Chaos in a periodically forced predator-prey ecosystem model.
*Mathematical Biosciences*, 113(1):91–113, 1993.

[2] Wikipedia contributors.
Lyapunov exponent, 2015.

[3] James MacDonald.
The damped driven pendulum: A chaotic system, 2013.